User's Guide

to

the PARI library

(version 2.7.6)

The PARI Group

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Chapter 4:
Programming PARI in Library Mode

The *User's Guide to Pari/GP* gives in three chapters a general presentation of the system, of the `gp` calculator, and detailed explanation of high level PARI routines available through the calculator. The present manual assumes general familiarity with the contents of these chapters and the basics of ANSI C programming, and focuses on the usage of the PARI library. In this chapter, we introduce the general concepts of PARI programming and describe useful general purpose functions; the following chapters describes all public low or high-level functions, underlying or extending the GP functions seen in Chapter 3 of the User's guide.

4.1 Introduction: initializations, universal objects.

To use PARI in library mode, you must write a C program and link it to the PARI library. See the installation guide or the Appendix to the *User's Guide to Pari/GP* on how to create and install the library and include files. A sample Makefile is presented in Appendix A, and a more elaborate one in `examples/Makefile`. The best way to understand how programming is done is to work through a complete example. We will write such a program in Section 4.10. Before doing this, a few explanations are in order.

First, one must explain to the outside world what kind of objects and routines we are going to use. This is done* with the directive

```
#include <pari/pari.h>
```

In particular, this defines the fundamental type for all PARI objects: the type **GEN**, which is simply a pointer to `long`.

Before any PARI routine is called, one must initialize the system, and in particular the PARI stack which is both a scratchboard and a repository for computed objects. This is done with a call to the function

```
void pari_init(size_t size, ulong maxprime)
```

The first argument is the number of bytes given to PARI to work with, and the second is the upper limit on a precomputed prime number table; `size` should not reasonably be taken below 500000 but you may set `maxprime` $= 0$, although the system still needs to precompute all primes up to about 2^{16}. For lower-level variants allowing finer control, e.g. preventing PARI from installing its own error or signal handlers, see Section 5.1.2.

We have now at our disposal:

• a PARI *stack* containing nothing. This is a big connected chunk of `size` bytes of memory, where all computations take place. In large computations, intermediate results quickly clutter up memory so some kind of garbage collecting is needed. Most systems do garbage collecting when the memory is getting scarce, and this slows down the performance. PARI takes a different approach,

* This assumes that PARI headers are installed in a directory which belongs to your compiler's search path for header files. You might need to add flags like `-I/usr/local/include` or modify `C_INCLUDE_PATH`.

admittedly more demanding on the programmer: you must do your own cleaning up when the intermediate results are not needed anymore. We will see later how (and when) this is done.

• the following *universal objects* (by definition, objects which do not belong to the stack): the integers 0, 1, −1, 2 and −2 (respectively called `gen_0`, `gen_1`, `gen_m1`, `gen_2` and `gen_m2`), the fraction $\frac{1}{2}$ (`ghalf`). All of these are of type `GEN`.

• a *heap* which is just a linked list of permanent universal objects. For now, it contains exactly the ones listed above. You will probably very rarely use the heap yourself; and if so, only as a collection of copies of objects taken from the stack (called clones in the sequel). Thus you need not bother with its internal structure, which may change as PARI evolves. Some complex PARI functions create clones for special garbage collecting purposes, usually destroying them when returning.

• a table of primes (in fact of *differences* between consecutive primes), called **diffptr**, of type `byteptr` (pointer to `unsigned char`). Its use is described in Section 5.4 later. Using it directly is deprecated, high-level iterators provide a cleaner and more flexible interface, see Section 4.8.2 (such iterators use the private prime table, but extend it dynamically).

• access to all the built-in functions of the PARI library. These are declared to the outside world when you include `pari.h`, but need the above things to function properly. So if you forget the call to `pari_init`, you will get a fatal error when running your program.

4.2 Important technical notes.

4.2.1 Backward compatibility. The PARI function names evolved over time, and deprecated functions are eventually deleted. The file `pariold.h` contains macros implementing a weak form of backward compatibility. In particular, whenever the name of a documented function changes, a `#define` is added to this file so that the old name expands to the new one (provided the prototype didn't change also).

This file is included by `pari.h`, but a large section is commented out by default. Define `PARI_OLD_NAMES` before including `pari.h` to pollute your namespace with lots of obsolete names like `un`*: that might enable you to compile old programs without having to modify them. The preferred way to do that is to add `-DPARI_OLD_NAMES` to your compiler `CFLAGS`, so that you don't need to modify the program files themselves.

Of course, it's better to fix the program if you can!

4.2.2 Types.

Although PARI objects all have the C type `GEN`, we will freely use the word **type** to refer to PARI dynamic subtypes: `t_INT`, `t_REAL`, etc. The declaration

```
GEN x;
```

declares a C variable of type `GEN`, but its "value" will be said to have type `t_INT`, `t_REAL`, etc. The meaning should always be clear from the context.

* For `(long)gen_1`. Since 2004 and version 2.2.9, typecasts are completely unnecessary in PARI programs.

4.2.3 Type recursivity.

Conceptually, most PARI types are recursive. But the `GEN` type is a pointer to `long`, not to `GEN`. So special macros must be used to access `GEN`'s components. The simplest one is `gel`(V, i), where **el** stands for **el**ement, to access component number i of the `GEN` V. This is a valid `lvalue` (may be put on the left side of an assignment), and the following two constructions are exceedingly frequent

```
gel(V, i) = x;
x = gel(V, i);
```

where `x` and `V` are `GEN`s. This macro accesses and modifies directly the components of V and do not create a copy of the coefficient, contrary to all the library *functions*.

More generally, to retrieve the values of elements of lists of ... of lists of vectors we have the `gmael` macros (for **m**ultidimensional **a**rray **el**ement). The syntax is **gmael**$n(V, a_1, \ldots, a_n)$, where V is a `GEN`, the a_i are indexes, and n is an integer between 1 and 5. This stands for $x[a_1][a_2]\ldots[a_n]$, and returns a `GEN`. The macros `gel` (resp. `gmael`) are synonyms for `gmael1` (resp. `gmael2`).

Finally, the macro `gcoeff`(M, i, j) has exactly the meaning of `M[i,j]` in GP when `M` is a matrix. Note that due to the implementation of `t_MAT`s as horizontal lists of vertical vectors, `gcoeff(x,y)` is actually equivalent to `gmael(y,x)`. One should use `gcoeff` in matrix context, and `gmael` otherwise.

4.2.4 Variations on basic functions. In the library syntax descriptions in Chapter 3, we have only given the basic names of the functions. For example `gadd`(x, y) assumes that x and y are `GEN`s, and *creates* the result $x+y$ on the PARI stack. For most of the basic operators and functions, many other variants are available. We give some examples for `gadd`, but the same is true for all the basic operators, as well as for some simple common functions (a complete list is given in Chapter 6):

```
GEN gaddgs(GEN x, long y)
```

```
GEN gaddsg(long x, GEN y)
```

In the following one, `z` is a preexisting `GEN` and the result of the corresponding operation is put into `z`. The size of the PARI stack does not change:

```
void gaddz(GEN x, GEN y, GEN z)
```

(This last form is inefficient in general and deprecated outside of PARI kernel programming.) Low level kernel functions implement these operators for specialized arguments and are also available: Level 0 deals with operations at the word level (`long`s and `ulong`s), Level 1 with `t_INT` and `t_REAL` and Level 2 with the rest (modular arithmetic, polynomial arithmetic and linear algebra). Here are some examples of Level 1 functions:

`GEN addii(GEN x, GEN y)`: here x and y are `GEN`s of type `t_INT` (this is not checked).

`GEN addrr(GEN x, GEN y)`: here x and y are `GEN`s of type `t_REAL` (this is not checked).

There also exist functions **addir**, **addri**, **mpadd** (whose two arguments can be of type `t_INT` or `t_REAL`), **addis** (to add a `t_INT` and a `long`) and so on.

The Level 1 names are self-explanatory once you know that **i** stands for a `t_INT`, **r** for a `t_REAL`, **mp** for i or r, **s** for a signed C long integer, **u** for an unsigned C long integer; finally the suffix **z** means that the result is not created on the PARI stack but assigned to a preexisting GEN object passed as an extra argument. Chapter 6 gives a description of these low-level functions.

Level 2 names are more complicated, see Section 7.1 for all the gory details, and we content ourselves with a simple example used to implement `t_INTMOD` arithmetic:

`GEN Fp_add(GEN x, GEN y, GEN m)`: returns the sum of x and y modulo m. Here x, y, m are `t_INT`s (this is not checked). The operation is more efficient if the inputs x, y are reduced modulo m, but this is not a necessary condition.

Important Note. These specialized functions are of course more efficient than the generic ones, but note the hidden danger here: the types of the objects involved (which is not checked) must be severely controlled, e.g. using `addii` on a `t_FRAC` argument will cause disasters. Type mismatches may corrupt the PARI stack, though in most cases they will just immediately overflow the stack. Because of this, the PARI philosophy of giving a result which is as exact as possible, enforced for generic functions like `gadd` or `gmul`, is dropped in kernel routines of Level 1, where it is replaced by the much simpler rule: the result is a `t_INT` if and only if all arguments are integer types (`t_INT` but also C `long` and `ulong`) and a `t_REAL` otherwise. For instance, multiplying a `t_REAL` by a `t_INT` always yields a `t_REAL` if you use `mulir`, where `gmul` returns the `t_INT` `gen_0` if the integer is 0.

4.2.5 Portability: 32-bit / 64-bit architectures.

PARI supports both 32-bit and 64-bit based machines, but not simultaneously! The library is compiled assuming a given architecture, and some of the header files you include (through `pari.h`) will have been modified to match the library.

Portable macros are defined to bypass most machine dependencies. If you want your programs to run identically on 32-bit and 64-bit machines, you have to use these, and not the corresponding numeric values, whenever the precise size of your `long` integers might matter. Here are the most important ones:

	64-bit	32-bit	
`BITS_IN_LONG`	64	32	
`LONG_IS_64BIT`	defined	undefined	
`DEFAULTPREC`	3	4	($\approx$ 19 decimal digits, see formula below)
`MEDDEFAULTPREC`	4	6	($\approx$ 38 decimal digits)
`BIGDEFAULTPREC`	5	8	($\approx$ 57 decimal digits)

For instance, suppose you call a transcendental function, such as

`GEN` **`gexp`**`(GEN x, long prec)`.

The last argument `prec` is an integer ≥ 3, corresponding to the default floating point precision required. It is *only* used if `x` is an exact object, otherwise the relative precision is determined by the precision of `x`. Since the parameter `prec` sets the size of the inexact result counted in (`long`) *words* (including codewords), the same value of `prec` will yield different results on 32-bit and 64-bit machines. Real numbers have two codewords (see Section 4.5), so the formula for computing the bit accuracy is

$$\texttt{bit\_accuracy}(\texttt{prec}) = (\texttt{prec} - 2) * \texttt{BITS\_IN\_LONG}$$

(this is actually the definition of an inline function). The corresponding accuracy expressed in decimal digits would be

$$\texttt{bit\_accuracy}(\texttt{prec}) * \log(2)/\log(10).$$

For example if the value of `prec` is 5, the corresponding accuracy for 32-bit machines is $(5-2) * \log(2^{32})/\log(10) \approx 28$ decimal digits, while for 64-bit machines it is $(5-2) * \log(2^{64})/\log(10) \approx 57$ decimal digits.

Thus, you must take care to change the `prec` parameter you are supplying according to the bit size, either using the default precisions given by the various `DEFAULTPREC`s, or by using conditional constructs of the form:

```
#ifndef LONG_IS_64BIT
  prec = 4;
#else
  prec = 6;
#endif
```

which is in this case equivalent to the statement `prec = MEDDEFAULTPREC;`.

Note that for parity reasons, half the accuracies available on 32-bit architectures (the odd ones) have no precise equivalents on 64-bit machines.

4.2.6 Using `malloc / free`. You should make use of the PARI stack as much as possible, and avoid allocating objects using the customary functions. If you do, you should use, or at least have a very close look at, the following wrappers:

`void* pari_malloc(size_t size)` calls `malloc` to allocate `size` bytes and returns a pointer to the allocated memory. If the request fails, an error is raised. The `SIGINT` signal is blocked until `malloc` returns, to avoid leaving the system stack in an inconsistent state.

`void* pari_realloc(void* ptr, size_t size)` as `pari_malloc` but calls `realloc` instead of `malloc`.

`void* pari_calloc(size_t size)` as `pari_malloc`, setting the memory to zero.

`void pari_free(void* ptr)` calls `free` to liberate the memory space pointed to by `ptr`, which must have been allocated by `malloc` (`pari_malloc`) or `realloc` (`pari_realloc`). The `SIGINT` signal is blocked until `free` returns.

If you use the standard `libc` functions instead of our wrappers, then your functions will be subtly incompatible with the `gp` calculator: when the user tries to interrupt a computation, the calculator may crash (if a system call is interrupted at the wrong time).

4.3 Garbage collection.

4.3.1 Why and how.

As we have seen, `pari_init` allocates a big range of addresses, the *stack*, that are going to be used throughout. Recall that all PARI objects are pointers. Except for a few universal objects, they all point at some part of the stack.

The stack starts at the address `bot` and ends just before `top`. This means that the quantity

$$(\texttt{top} - \texttt{bot}) \,/\, \texttt{sizeof(long)}$$

is (roughly) equal to the `size` argument of `pari_init`. The PARI stack also has a "current stack pointer" called **avma**, which stands for **av**ailable **m**emory **a**ddress. These three variables are global (declared by `pari.h`). They are of type `pari_sp`, which means *pari stack pointer*.

The stack is oriented upside-down: the more recent an object, the closer to `bot`. Accordingly, initially `avma` = `top`, and `avma` gets *decremented* as new objects are created. As its name indicates,

`avma` always points just *after* the first free address on the stack, and `(GEN)avma` is always (a pointer to) the latest created object. When `avma` reaches `bot`, the stack overflows, aborting all computations, and an error message is issued. To avoid this *you* need to clean up the stack from time to time, when intermediate objects are not needed anymore. This is called "*garbage collecting*."

We are now going to describe briefly how this is done. We will see many concrete examples in the next subsection.

• First, PARI routines do their own garbage collecting, which means that whenever a documented function from the library returns, only its result(s) have been added to the stack, possibly up to a very small overhead (non-documented ones may not do this). In particular, a PARI function that does not return a `GEN` does not clutter the stack. Thus, if your computation is small enough (e.g. you call few PARI routines, or most of them return `long` integers), then you do not need to do any garbage collecting. This is probably the case in many of your subroutines. Of course the objects that were on the stack *before* the function call are left alone. Except for the ones listed below, PARI functions only collect their own garbage.

• It may happen that all objects that were created after a certain point can be deleted — for instance, if the final result you need is not a `GEN`, or if some search proved futile. Then, it is enough to record the value of `avma` just *before* the first garbage is created, and restore it upon exit:

```
pari_sp av = avma; /* record initial avma */

garbage ...
avma = av; /* restore it */
```

All objects created in the `garbage` zone will eventually be overwritten: they should no longer be accessed after `avma` has been restored.

• If you want to destroy (i.e. give back the memory occupied by) the *latest* PARI object on the stack (e.g. the latest one obtained from a function call), you can use the function

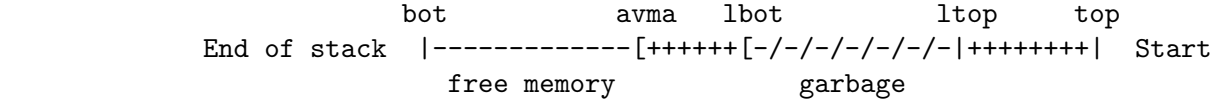
```
void cgiv(GEN z)
```

where `z` is the object you want to give back. This is equivalent to the above where the initial `av` is computed from `z`.

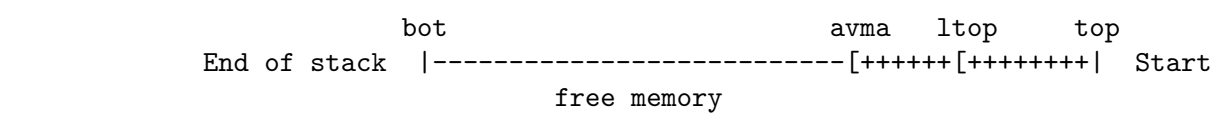
• Unfortunately life is not so simple, and sometimes you will want to give back accumulated garbage *during* a computation without losing recent data. We shall start with the lowest level function to get a feel for the underlying mechanisms, we shall describe simpler variants later:

`GEN gerepile(pari_sp ltop, pari_sp lbot, GEN q)`. This function cleans up the stack between `ltop` and `lbot`, where `lbot` $<$ `ltop`, and returns the updated object `q`. This means:

1) we translate (copy) all the objects in the interval [`avma`, `lbot`[, so that its right extremity abuts the address `ltop`. Graphically



becomes:



where ++ denote significant objects, -- the unused part of the stack, and -/- the garbage we remove.

2) The function then inspects all the PARI objects between `avma` and `lbot` (i.e. the ones that we want to keep and that have been translated) and looks at every component of such an object which is not a codeword. Each such component is a pointer to an object whose address is either

— between `avma` and `lbot`, in which case it is suitably updated,

— larger than or equal to `ltop`, in which case it does not change, or

— between `lbot` and `ltop` in which case `gerepile` raises an error ("significant pointers lost in gerepile").

3) **avma** is updated (we add `ltop` − `lbot` to the old value).

4) We return the (possibly updated) object `q`: if `q` initially pointed between `avma` and `lbot`, we return the updated address, as in 2). If not, the original address is still valid, and is returned!

As stated above, no component of the remaining objects (in particular `q`) should belong to the erased segment [`lbot`, `ltop`[, and this is checked within `gerepile`. But beware as well that the addresses of the objects in the translated zone change after a call to `gerepile`, so you must not access any pointer which previously pointed into the zone below `ltop`. If you need to recover more than one object, use the `gerepileall` function below.

Remark. As a consequence of the preceding explanation, if a PARI object is to be relocated by gerepile then, apart from universal objects, the chunks of memory used by its components should be in consecutive memory locations. All `GEN`s created by documented PARI functions are guaranteed to satisfy this. This is because the `gerepile` function knows only about *two connected zones*: the garbage that is erased (between `lbot` and `ltop`) and the significant pointers that are copied and updated. If there is garbage interspersed with your objects, disaster occurs when we try to update them and consider the corresponding "pointers". In most cases of course the said garbage is in fact a bunch of other `GEN`s, in which case we simply waste time copying and updating them for nothing. But be wary when you allow objects to become disconnected.

In practice this is achieved by the following programming idiom:

```
ltop = avma; garbage(); lbot = avma; q = anything();
return gerepile(ltop, lbot, q); /* returns the updated q */
```

or directly

```
ltop = avma; garbage(); lbot = avma;
return gerepile(ltop, lbot, anything());
```

Beware that

```
ltop = avma; garbage();
return gerepile(ltop, avma, anything())
```

might work, but should be frowned upon. We cannot predict whether `avma` is evaluated after or before the call to `anything()`: it depends on the compiler. If we are out of luck, it is *after* the call, so the result belongs to the garbage zone and the `gerepile` statement becomes equivalent to `avma = ltop`. Thus we return a pointer to random garbage.

4.3.2 Variants.

`GEN gerepileupto(pari_sp ltop, GEN q)`. Cleans the stack between `ltop` and the *connected* object `q` and returns `q` updated. For this to work, `q` must have been created *before* all its components, otherwise they would belong to the garbage zone! Unless mentioned otherwise, documented PARI functions guarantee this.

`GEN gerepilecopy(pari_sp ltop, GEN x)`. Functionally equivalent to, but more efficient than

```
gerepileupto(ltop, gcopy(x))
```

In this case, the `GEN` parameter `x` need not satisfy any property before the garbage collection: it may be disconnected, components created before the root, and so on. Of course, this is about twice slower than either `gerepileupto` or `gerepile`, because `x` has to be copied to a clean stack zone first. This function is a special case of `gerepileall` below, where $n = 1$.

`void gerepileall(pari_sp ltop, int n, ...)`. To cope with complicated cases where many objects have to be preserved. The routine expects n further arguments, which are the *addresses* of the `GEN`s you want to preserve:

```
pari_sp ltop = avma;
...; y = ...; ... x = ...; ...;
gerepileall(ltop, 2, &x, &y);
```

It cleans up the most recent part of the stack (between `ltop` and `avma`), updating all the `GEN`s added to the argument list. A copy is done just before the cleaning to preserve them, so they do not need to be connected before the call. With `gerepilecopy`, this is the most robust of the `gerepile` functions (the less prone to user error), hence the slowest.

`void gerepileallsp(pari_sp ltop, pari_sp lbot, int n, ...)`. More efficient, but trickier than `gerepileall`. Cleans the stack between `lbot` and `ltop` and updates the `GEN`s pointed at by the elements of `gptr` without any further copying. This is subject to the same restrictions as `gerepile`, the only difference being that more than one address gets updated.

4.3.3 Examples.

4.3.3.1 gerepile.

Let `x` and `y` be two preexisting PARI objects and suppose that we want to compute $\mathtt{x}^2 + \mathtt{y}^2$. This is done using the following program:

```
GEN x2 = gsqr(x);
GEN y2 = gsqr(y), z = gadd(x2,y2);
```

The `GEN z` indeed points at the desired quantity. However, consider the stack: it contains as unnecessary garbage `x2` and `y2`. More precisely it contains (in this order) `z`, `y2`, `x2`. (Recall that, since the stack grows downward from the top, the most recent object comes first.)

It is not possible to get rid of `x2`, `y2` before `z` is computed, since they are used in the final operation. We cannot record `avma` before `x2` is computed and restore it later, since this would destroy `z` as well. It is not possible either to use the function `cgiv` since `x2` and `y2` are not at the bottom of the stack and we do not want to give back `z`.

But using `gerepile`, we can give back the memory locations corresponding to `x2`, `y2`, and move the object `z` upwards so that no space is lost. Specifically:

```
pari_sp ltop = avma;  /* remember the current top of the stack */
```

```
GEN x2 = gsqr(x);
GEN y2 = gsqr(y);
pari_sp lbot = avma;  /* the bottom of the garbage pile */
GEN z = gadd(x2, y2); /* z is now the last object on the stack */
z = gerepile(ltop, lbot, z);
```

Of course, the last two instructions could also have been written more simply:

```
z = gerepile(ltop, lbot, gadd(x2,y2));
```

In fact `gerepileupto` is even simpler to use, because the result of `gadd` is the last object on the stack and `gadd` is guaranteed to return an object suitable for `gerepileupto`:

```
ltop = avma;
z = gerepileupto(ltop, gadd(gsqr(x), gsqr(y)));
```

Make sure you understand exactly what has happened before you go on!

Remark on assignments and gerepile. When the tree structure and the size of the PARI objects which will appear in a computation are under control, one may allocate sufficiently large objects at the beginning, use assignment statements, then simply restore `avma`. Coming back to the above example, note that *if* we know that x and y are of type real fitting into `DEFAULTPREC` words, we can program without using `gerepile` at all:

```
z = cgetr(DEFAULTPREC); ltop = avma;
gaffect(gadd(gsqr(x), gsqr(y)), z);
avma = ltop;
```

This is often *slower* than a craftily used `gerepile` though, and certainly more cumbersome to use. As a rule, assignment statements should generally be avoided.

Variations on a theme. it is often necessary to do several `gerepile`s during a computation. However, the fewer the better. The only condition for `gerepile` to work is that the garbage be connected. If the computation can be arranged so that there is a minimal number of connected pieces of garbage, then it should be done that way.

For example suppose we want to write a function of two `GEN` variables `x` and `y` which creates the vector $[x^2 + y, y^2 + x]$. Without garbage collecting, one would write:

```
p1 = gsqr(x); p2 = gadd(p1, y);
p3 = gsqr(y); p4 = gadd(p3, x);
z = mkvec2(p2, p4);  /* not suitable for gerepileupto! */
```

This leaves a dirty stack containing (in this order) `z`, `p4`, `p3`, `p2`, `p1`. The garbage here consists of `p1` and `p3`, which are separated by `p2`. But if we compute `p3` *before* `p2` then the garbage becomes connected, and we get the following program with garbage collecting:

```
ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(p1,y);
gel(z, 2) = gadd(p3,x); z = gerepile(ltop,lbot,z);
```

Finishing by `z = gerepileupto(ltop, z)` would be ok as well. Beware that

```
ltop = avma; p1 = gadd(gsqr(x), y); p3 = gadd(gsqr(y), x);
z = cgetg(3, t_VEC);
```

```
gel(z, 1) = p1;
gel(z, 2) = p3; z = gerepileupto(ltop,z); /* WRONG */
```

is a disaster since `p1` and `p3` are created before `z`, so the call to `gerepileupto` overwrites them, leaving `gel(z, 1)` and `gel(z, 2)` pointing at random data! The following does work:

```
ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = mkvec2(gadd(p1,y), gadd(p3,x));
z = gerepile(ltop,lbot,z);
```

but is very subtly wrong in the sense that `z = gerepileupto(ltop, z)` would *not* work. The reason being that `mkvec2` creates the root `z` of the vector *after* its arguments have been evaluated, creating the components of `z` too early; `gerepile` does not care, but the created `z` is a time bomb which will explode on any later `gerepileupto`. On the other hand

```
ltop = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(gsqr(x), y);
gel(z, 2) = gadd(gsqr(y), x); z = gerepileupto(ltop,z); /* INEFFICIENT */
```

leaves the results of `gsqr(x)` and `gsqr(y)` on the stack (and lets `gerepileupto` update them for naught). Finally, the most elegant and efficient version (with respect to time and memory use) is as follows

```
z = cgetg(3, t_VEC);
ltop = avma; gel(z, 1) = gerepileupto(ltop, gadd(gsqr(x), y));
ltop = avma; gel(z, 2) = gerepileupto(ltop, gadd(gsqr(y), x));
```

which avoids updating the container `z` and cleans up its components individually, as soon as they are computed.

One last example. Let us compute the product of two complex numbers x and y, using the $3M$ method which requires 3 multiplications instead of the obvious 4. Let $z = x*y$, and set $x = x_r + i*x_i$ and similarly for y and z. We compute $p_1 = x_r * y_r$, $p_2 = x_i * y_i$, $p_3 = (x_r + x_i) * (y_r + y_i)$, and then we have $z_r = p_1 - p_2$, $z_i = p_3 - (p_1 + p_2)$. The program is as follows:

```
ltop = avma;
p1 = gmul(gel(x,1), gel(y,1));
p2 = gmul(gel(x,2), gel(y,2));
p3 = gmul(gadd(gel(x,1), gel(x,2)), gadd(gel(y,1), gel(y,2)));
p4 = gadd(p1,p2);
lbot = avma; z = cgetg(3, t_COMPLEX);
gel(z, 1) = gsub(p1,p2);
gel(z, 2) = gsub(p3,p4); z = gerepile(ltop,lbot,z);
```

Exercise. Write a function which multiplies a matrix by a column vector. Hint: start with a `cgetg` of the result, and use `gerepile` whenever a coefficient of the result vector is computed. You can look at the answer in `src/basemath/RgV.c:RgM_RgC_mul()`.

4.3.3.2 gerepileall.

Let us now see why we may need the `gerepileall` variants. Although it is not an infrequent occurrence, we do not give a specific example but a general one: suppose that we want to do a computation (usually inside a larger function) producing more than one PARI object as a result, say two for instance. Then even if we set up the work properly, before cleaning up we have a stack which has the desired results `z1`, `z2` (say), and then connected garbage from lbot to ltop. If we write

```
z1 = gerepile(ltop, lbot, z1);
```

then the stack is cleaned, the pointers fixed up, but we have lost the address of `z2`. This is where we need the gerepileall function:

```
gerepileall(ltop, 2, &z1, &z2)
```

copies `z1` and `z2` to new locations, cleans the stack from `ltop` to the old `avma`, and updates the pointers `z1` and `z2`. Here we do not assume anything about the stack: the garbage can be disconnected and `z1`, `z2` need not be at the bottom of the stack. If all of these assumptions are in fact satisfied, then we can call `gerepilemanysp` instead, which is usually faster since we do not need the initial copy (on the other hand, it is less cache friendly).

A most important usage is "random" garbage collection during loops whose size requirements we cannot (or do not bother to) control in advance:

```
pari_sp ltop = avma, limit = stack_lim(avma, 1);
GEN x, y;
while (...)
{
  garbage(); x = anything();
  garbage(); y = anything(); garbage();
  if (avma < limit) /* memory is running low (half spent since entry) */
    gerepileall(ltop, 2, &x, &y);
}
```

Here we assume that only `x` and `y` are needed from one iteration to the next. As it would be costly to call gerepile once for each iteration, we only do it when it seems to have become necessary. The macro `stack_lim(avma,`n`)` denotes an address where $2^{n-1}/(2^{n-1}+1)$ of the remaining stack space is exhausted (1/2 for $n = 1$, 2/3 for $n = 2$).

4.3.4 Comments.

First, `gerepile` has turned out to be a flexible and fast garbage collector for number-theoretic computations, which compares favorably with more sophisticated methods used in other systems. Our benchmarks indicate that the price paid for using `gerepile` and `gerepile`-related copies, when properly used, is usually less than 1% of the total running time, which is quite acceptable!

Second, it is of course harder on the programmer, and quite error-prone if you do not stick to a consistent PARI programming style. If all seems lost, just use `gerepilecopy` (or `gerepileall`) to fix up the stack for you. You can always optimize later when you have sorted out exactly which routines are crucial and what objects need to be preserved and their usual sizes.

If you followed us this far, congratulations, and rejoice: the rest is much easier.

4.4 Creation of PARI objects, assignments, conversions.

4.4.1 Creation of PARI objects. The basic function which creates a PARI object is

`GEN cgetg(long l, long t)` l specifies the number of longwords to be allocated to the object, and t is the type of the object, in symbolic form (see Section 4.5 for the list of these). The precise effect of this function is as follows: it first creates on the PARI *stack* a chunk of memory of size `length` longwords, and saves the address of the chunk which it will in the end return. If the stack has been used up, a message to the effect that "the PARI stack overflows" is printed, and an error raised. Otherwise, it sets the type and length of the PARI object. In effect, it fills its first codeword (`z[0]`). Many PARI objects also have a second codeword (types `t_INT`, `t_REAL`, `t_PADIC`, `t_POL`, and `t_SER`). In case you want to produce one of those from scratch, which should be exceedingly rare, *it is your responsibility to fill this second codeword*, either explicitly (using the macros described in Section 4.5), or implicitly using an assignment statement (using `gaffect`).

Note that the length argument l is predetermined for a number of types: 3 for types `t_INTMOD`, `t_FRAC`, `t_COMPLEX`, `t_POLMOD`, `t_RFRAC`, 4 for type `t_QUAD` and `t_QFI`, and 5 for type `t_PADIC` and `t_QFR`. However for the sake of efficiency, `cgetg` does not check this: disasters will occur if you give an incorrect length for those types.

Notes. 1) The main use of this function is create efficiently a constant object, or to prepare for later assignments (see Section 4.4.3). Most of the time you will use `GEN` objects as they are created and returned by PARI functions. In this case you do not need to use `cgetg` to create space to hold them.

2) For the creation of leaves, i.e. `t_INT` or `t_REAL`,

```
GEN cgeti(long length)
```

```
GEN cgetr(long length)
```

should be used instead of `cgetg(length, t_INT)` and `cgetg(length, t_REAL)` respectively. Finally

```
GEN cgetc(long prec)
```

creates a `t_COMPLEX` whose real and imaginary part are `t_REAL`s allocated by `cgetr(prec)`.

Examples. 1) Both `z = cgeti(DEFAULTPREC)` and `cgetg(DEFAULTPREC, t_INT)` create a `t_INT` whose "precision" is `bit_accuracy(DEFAULTPREC)` = 64. This means `z` can hold rational integers of absolute value less than 2^{64}. Note that in both cases, the second codeword is *not* filled. Of course we could use numerical values, e.g. `cgeti(4)`, but this would have different meanings on different machines as `bit_accuracy(4)` equals 64 on 32-bit machines, but 128 on 64-bit machines.

2) The following creates a *complex number* whose real and imaginary parts can hold real numbers of precision `bit_accuracy(MEDDEFAULTPREC)` = 96 bits:

```
z = cgetg(3, t_COMPLEX);
gel(z, 1) = cgetr(MEDDEFAULTPREC);
gel(z, 2) = cgetr(MEDDEFAULTPREC);
```

or simply `z = cgetc(MEDDEFAULTPREC)`.

3) To create a matrix object for 4×3 matrices:

```
z = cgetg(4, t_MAT);
for(i=1; i<4; i++) gel(z, i) = cgetg(5, t_COL);
```

or simply `z = zeromatcopy(4, 3)`, which further initializes all entries to `gen_0`.

These last two examples illustrate the fact that since PARI types are recursive, all the branches of the tree must be created. The function **cgetg** creates only the "root", and other calls to `cgetg` must be made to produce the whole tree. For matrices, a common mistake is to think that `z = cgetg(4, t_MAT)` (for example) creates the root of the matrix: one needs also to create the column vectors of the matrix (obviously, since we specified only one dimension in the first `cgetg`!). This is because a matrix is really just a row vector of column vectors (hence a priori not a basic type), but it has been given a special type number so that operations with matrices become possible.

Finally, to facilitate input of constant objects when speed is not paramount, there are four `varargs` functions:

`GEN mkintn(long n, ...)` returns the non-negative `t_INT` whose development in base 2^{32} is given by the following n words (`unsigned long`). It is assumed that all such arguments are less than 2^{32} (the actual `sizeof(long)` is irrelevant, the behavior is also as above on 64-bit machines).

```
mkintn(3, a2, a1, a0);
```

returns $a_2 2^{64} + a_1 2^{32} + a_0$.

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose n coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

Warning. Contrary to the policy of general PARI functions, the latter three functions do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. For instance

```
/* gerepile-safe: components are universal objects */
z = mkvecn(3, gen_1, gen_0, gen_2);

/* not OK for gerepileupto: stoi(3) creates component before root */
z = mkvecn(3, stoi(3), gen_0, gen_2);

/* NO! First vector component x is destroyed */
x = gclone(gen_1);
z = mkvecn(3, x, gen_0, gen_2);
gunclone(x);
```

The following function is also available as a special case of `mkintn`:

`GEN uu32toi(ulong a, ulong b)`

Returns the `GEN` equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

4.4.2 Sizes.

`long gsizeword(GEN x)` returns the total number of `BITS_IN_LONG`-bit words occupied by the tree representing `x`.

`long gsizebyte(GEN x)` returns the total number of bytes occupied by the tree representing `x`, i.e. `gsizeword(x)` multiplied by `sizeof(long)`. This is normally useless since PARI functions use a number of *words* as input for lengths and precisions.

4.4.3 Assignments. Firstly, if `x` and `y` are both declared as `GEN` (i.e. pointers to something), the ordinary C assignment `y = x` makes perfect sense: we are just moving a pointer around. However, physically modifying either `x` or `y` (for instance, `x[1] = 0`) also changes the other one, which is usually not desirable.

Very important note. Using the functions described in this paragraph is inefficient and often awkward: one of the `gerepile` functions (see Section 4.3) should be preferred. See the paragraph end for one exception to this rule.

The general PARI assignment function is the function **gaffect** with the following syntax:

`void gaffect(GEN x, GEN y)`

Its effect is to assign the PARI object `x` into the *preexisting* object `y`. Both `x` and `y` must be *scalar* types. For convenience, vector or matrices of scalar types are also allowed.

This copies the whole structure of `x` into `y` so many conditions must be met for the assignment to be possible. For instance it is allowed to assign a `t_INT` into a `t_REAL`, but the converse is forbidden. For that, you must use the truncation or rounding function of your choice, e.g.`mpfloor`.

It can also happen that `y` is not large enough or does not have the proper tree structure to receive the object `x`. For instance, let `y` the zero integer with length equal to 2; then `y` is too small to accommodate any non-zero `t_INT`. In general common sense tells you what is possible, keeping in mind the PARI philosophy which says that if it makes sense it is valid. For instance, the assignment of an imprecise object into a precise one does *not* make sense. However, a change in precision of imprecise objects is allowed, even if it *increases* its accuracy: we complement the

"mantissa" with infinitely many 0 digits in this case. (Mantissa between quotes, because this is not restricted to `t_REAL`s, it also applies for p-adics for instance.)

All functions ending in "**z**" such as **gaddz** (see Section 4.2.4) implicitly use this function. In fact what they exactly do is record **avma** (see Section 4.3), perform the required operation, **gaffect** the result to the last operand, then restore the initial `avma`.

You can assign ordinary C long integers into a PARI object (not necessarily of type `t_INT`) using

```
void gaffsg(long s, GEN y)
```

Note. Due to the requirements mentioned above, it is usually a bad idea to use `gaffect` statements. There is one exception: for simple objects (e.g. leaves) whose size is controlled, they can be easier to use than `gerepile`, and about as efficient.

Coercion. It is often useful to coerce an inexact object to a given precision. For instance at the beginning of a routine where precision can be kept to a minimum; otherwise the precision of the input is used in all subsequent computations, which is inefficient if the latter is known to thousands of digits. One may use the `gaffect` function for this, but it is easier and more efficient to call

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of length `prec`.

4.4.4 Copy. It is also very useful to copy a PARI object, not just by moving around a pointer as in the `y = x` example, but by creating a copy of the whole tree structure, without pre-allocating a possibly complicated `y` to use with `gaffect`. The function which does this is called **gcopy**. Its syntax is:

```
GEN gcopy(GEN x)
```

and the effect is to create a new copy of x on the PARI stack.

Sometimes, on the contrary, a quick copy of the skeleton of `x` is enough, leaving pointers to the original data in `x` for the sake of speed instead of making a full recursive copy. Use `GEN shallowcopy(GEN x)` for this. Note that the result is not suitable for `gerepileupto` !

Make sure at this point that you understand the difference between `y = x`, `y = gcopy(x)`, `y = shallowcopy(x)` and `gaffect(x,y)`.

4.4.5 Clones. Sometimes, it is more efficient to create a *persistent* copy of a PARI object. This is not created on the stack but on the heap, hence unaffected by `gerepile` and friends. The function which does this is called **gclone**. Its syntax is:

```
GEN gclone(GEN x)
```

A clone can be removed from the heap (thus destroyed) using

```
void gunclone(GEN x)
```

No PARI object should keep references to a clone which has been destroyed!

4.4.6 Conversions. The following functions convert C objects to PARI objects (creating them on the stack as usual):

`GEN stoi(long s)`: C `long` integer ("small") to `t_INT`.

`GEN dbltor(double s)`: C `double` to `t_REAL`. The accuracy of the result is 19 decimal digits, i.e. a type `t_REAL` of length `DEFAULTPREC`, although on 32-bit machines only 16 of them are significant.

We also have the converse functions:

`long itos(GEN x)`: `x` must be of type `t_INT`,

`double rtodbl(GEN x)`: `x` must be of type `t_REAL`,

as well as the more general ones:

`long gtolong(GEN x)`,

`double gtodouble(GEN x)`.

4.5 Implementation of the PARI types.

We now go through each type and explain its implementation. Let `z` be a `GEN`, pointing at a PARI object. In the following paragraphs, we will constantly mix two points of view: on the one hand, `z` is treated as the C pointer it is, on the other, as PARI's handle on some mathematical entity, so we will shamelessly write $\texttt{z} \neq 0$ to indicate that the *value* thus represented is nonzero (in which case the *pointer* `z` is certainly non-`NULL`). We offer no apologies for this style. In fact, you had better feel comfortable juggling both views simultaneously in your mind if you want to write correct PARI programs.

Common to all the types is the first codeword `z[0]`, which we do not have to worry about since this is taken care of by `cgetg`. Its precise structure depends on the machine you are using, but it always contains the following data: the *internal type number* associated to the symbolic type name, the *length* of the root in longwords, and a technical bit which indicates whether the object is a clone or not (see Section 4.4.5). This last one is used by `gp` for internal garbage collecting, you will not have to worry about it.

Some types have a second codeword, different for each type, which we will soon describe as we will shortly consider each of them in turn.

The first codeword is handled through the following *macros*:

`long typ(GEN z)` returns the type number of `z`.

`void settyp(GEN z, long n)` sets the type number of `z` to `n` (you should not have to use this function if you use `cgetg`).

`long lg(GEN z)` returns the length (in longwords) of the root of `z`.

`long setlg(GEN z, long l)` sets the length of `z` to `l` (you should not have to use this function if you use `cgetg`; however, see an advanced example in Section 4.10).

`long isclone(GEN z)` is `z` a clone?

`void setisclone(GEN z)` sets the *clone* bit.

`void unsetisclone(GEN z)` clears the *clone* bit.

Important remark. For the sake of efficiency, none of the codeword-handling macros check the types of their arguments even when there are stringent restrictions on their use. It is trivial to create invalid objects, or corrupt one of the "universal constants" (e.g. setting the sign of `gen_0` to 1), and they usually provide negligible savings. Use higher level functions whenever possible.

Remark. The clone bit is there so that `gunclone` can check it is deleting an object which was allocated by `gclone`. Miscellaneous vector entries are often cloned by `gp` so that a GP statement like `v[1] = x` does not involve copying the whole of `v`: the component `v[1]` is deleted if its clone bit is set, and is replaced by a clone of `x`. Don't set/unset yourself the clone bit unless you know what you are doing: in particular *never* set the clone bit of a vector component when the said vector is scheduled to be uncloned. Hackish code may abuse the clone bit to tag objects for reasons unrelated to the above instead of using proper data structures. Don't do that.

4.5.1 Type `t_INT` (integer). this type has a second codeword `z[1]` which contains the following information:

the sign of `z`: coded as 1, 0 or -1 if $\mathtt{z} > 0$, $\mathtt{z} = 0$, $\mathtt{z} < 0$ respectively.

the *effective length* of `z`, i.e. the total number of significant longwords. This means the following: apart from the integer 0, every integer is "normalized", meaning that the most significant mantissa longword is non-zero. However, the integer may have been created with a longer length. Hence the "length" which is in `z[0]` can be larger than the "effective length" which is in `z[1]`.

This information is handled using the following macros:

`long signe(GEN z)` returns the sign of `z`.

`void setsigne(GEN z, long s)` sets the sign of `z` to `s`.

`long lgefint(GEN z)` returns the effective length of `z`.

`void setlgefint(GEN z, long l)` sets the effective length of `z` to `l`.

The integer 0 can be recognized either by its sign being 0, or by its effective length being equal to 2. Now assume that $\mathtt{z} \neq 0$, and let

$$|z| = \sum_{i=0}^{n} z_i B^i, \quad \text{where } z_n \neq 0 \text{ and } B = 2^{\mathtt{BITS\_IN\_LONG}}.$$

With these notations, n is `lgefint(z) - 3`, and the mantissa of `z` may be manipulated via the following interface:

`GEN int_MSW(GEN z)` returns a pointer to the most significant word of `z`, z_n.

`GEN int_LSW(GEN z)` returns a pointer to the least significant word of `z`, z_0.

`GEN int_W(GEN z, long i)` returns the i-th significant word of `z`, z_i. Accessing the i-th significant word for $i > n$ yields unpredictable results.

`GEN int_W_lg(GEN z, long i, long lz)` returns the i-th significant word of `z`, z_i, assuming `lgefint(z)` is `lz` ($= n + 3$). Accessing the i-th significant word for $i > n$ yields unpredictable results.

`GEN int_precW(GEN z)` returns the previous (less significant) word of `z`, z_{i-1} assuming `z` points to z_i.

`GEN int_nextW(GEN z)` returns the next (more significant) word of `z`, z_{i+1} assuming `z` points to z_i.

Unnormalized integers, such that z_n is possibly 0, are explicitly forbidden. To enforce this, one may write an arbitrary mantissa then call

`void int_normalize(GEN z, long known0)`

normalizes in place a non-negative integer (such that z_n is possibly 0), assuming at least the first `known0` words are zero.

For instance a binary `and` could be implemented in the following way:

```
GEN AND(GEN x, GEN y) {
  long i, lx, ly, lout;
  long *xp, *yp, *outp; /* mantissa pointers */
  GEN out;

  if (!signe(x) || !signe(y)) return gen_0;
  lx = lgefint(x); xp = int_LSW(x);
  ly = lgefint(y); yp = int_LSW(y); lout = min(lx,ly); /* > 2 */

  out = cgeti(lout); out[1] = evalsigne(1) | evallgefint(lout);
  outp = int_LSW(out);
  for (i=2; i < lout; i++)
  {
    *outp = (*xp) & (*yp);
    outp  = int_nextW(outp);
    xp    = int_nextW(xp);
    yp    = int_nextW(yp);
  }
  if ( !*int_MSW(out) ) out = int_normalize(out, 1);
  return out;
}
```

This low-level interface is mandatory in order to write portable code since PARI can be compiled using various multiprecision kernels, for instance the native one or GNU MP, with incompatible internal structures (for one thing, the mantissa is oriented in different directions).

The following further functions are available:

`int mpodd(GEN x)` which is 1 if `x` is odd, and 0 otherwise.

`long mod2(GEN x)`

`long mod4(GEN x)`

`long mod8(GEN x)`

`long mod16(GEN x)`

`long mod32(GEN x)`

`long mod64(GEN x)` give the residue class of `x` modulo the corresponding power of 2, for *positive* `x`. By definition, `mod`$n(x) :=$ `mod`$n(|x|)$ for $x < 0$ (the functions disregard the sign), and the result is undefined if $x = 0$. As well,

`ulong mod2BIL(GEN x)` returns the least significant word of $|x|$, still assuming that $x \neq 0$.

These functions directly access the binary data and are thus much faster than the generic modulo functions. Besides, they return long integers instead of GENs, so they do not clutter up the stack.

4.5.2 Type t_REAL (real number). this type has a second codeword z[1] which also encodes its sign, obtained or set using the same functions as for a t_INT, and a binary exponent. This exponent is handled using the following macros:

`long expo(GEN z)` returns the exponent of z. This is defined even when z is equal to zero, see Section ??.

`void setexpo(GEN z, long e)` sets the exponent of z to e.

Note the functions:

`long gexpo(GEN z)` which tries to return an exponent for z, even if z is not a real number.

`long gsigne(GEN z)` which returns a sign for z, even when z is neither real nor integer (a rational number for instance).

The real zero is characterized by having its sign equal to 0. If z is not equal to 0, then is is represented as $2^e M$, where e is the exponent, and $M \in [1, 2[$ is the mantissa of z, whose digits are stored in $\mathtt{z}[2], \ldots, \mathtt{z}[\mathtt{lg(z)} - 1]$.

More precisely, let m be the integer (z[2],..., z[lg(z)-1]) in base 2^BITS_IN_LONG; here, z[2] is the most significant longword and is normalized, i.e. its most significant bit is 1. Then we have $M := m/2^{\mathtt{bit\_accuracy(lg(z))}-1-\mathtt{expo}(z)}$.

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z, and sets *e to the exponent $\mathtt{bit\_accuracy(lg(z))} - 1 - \mathtt{expo}(z)$, so that $z = m/2^e$.

Thus, the real number 3.5 to accuracy bit_accuracy(lg(z)) is represented as z[0] (encoding type = t_REAL, lg(z)), z[1] (encoding sign = 1, expo = 1), z[2] = 0xe0000000, z[3] = ... = z[lg(z) − 1] = 0x0.

4.5.3 Type t_INTMOD. z[1] points to the modulus, and z[2] at the number representing the class z. Both are separate GEN objects, and both must be t_INTs, satisfying the inequality $0 \leq \mathtt{z}[2] < \mathtt{z}[1]$.

4.5.4 Type t_FRAC (rational number). z[1] points to the numerator n, and z[2] to the denominator d. Both must be of type t_INT such that $n \neq 0$, $d > 0$ and $(n, d) = 1$.

4.5.5 Type t_FFELT (finite field element). (Experimental)

Components of this type should normally not be accessed directly. Instead, finite field elements should be created using ffgen.

The second codeword z[1] determines the storage format of the element, among

• t_FF_FpXQ: A=z[2] and T=z[3] are FpX, p=z[4] is a t_INT, where p is a prime number, T is irreducible modulo p, and $\deg A < \deg T$. This represents the element $A \pmod T$ in $\mathbf{F}_p[X]/T$.

• t_FF_Flxq: A=z[2] and T=z[3] are Flx, l=z[4] is a t_INT, where l is a prime number, T is irreducible modulo l, and $\deg A < \deg T$ This represents the element $A \pmod T$ in $\mathbf{F}_l[X]/T$.

• t_FF_F2xq: A=z[2] and T=z[3] are F2x, l=z[4] is the t_INT 2, T is irreducible modulo 2, and $\deg A < \deg T$. This represents the element $A \pmod T$ in $\mathbf{F}_2[X]/T$.

4.5.6 Type `t_COMPLEX` (complex number). `z[1]` points to the real part, and `z[2]` to the imaginary part. The components `z[1]` and `z[2]` must be of type `t_INT`, `t_REAL` or `t_FRAC`. For historical reasons `t_INTMOD` and `t_PADIC` are also allowed (the latter for $p = 2$ or congruent to 3 mod 4 only), but one should rather use the more general `t_POLMOD` construction.

4.5.7 Type `t_PADIC` (p-adic numbers). this type has a second codeword `z[1]` which contains the following information: the p-adic precision (the exponent of p modulo which the p-adic unit corresponding to `z` is defined if `z` is not 0), i.e. one less than the number of significant p-adic digits, and the exponent of `z`. This information can be handled using the following functions:

`long precp(GEN z)` returns the p-adic precision of `z`. This is 0 if $\mathtt{z} = 0$.

`void setprecp(GEN z, long l)` sets the p-adic precision of `z` to `l`.

`long valp(GEN z)` returns the p-adic valuation of `z` (i.e. the exponent). This is defined even if `z` is equal to 0, see Section **??**.

`void setvalp(GEN z, long e)` sets the p-adic valuation of `z` to `e`.

In addition to this codeword, `z[2]` points to the prime p, `z[3]` points to $p^{\mathrm{precp(z)}}$, and `z[4]` points to a`t_INT` representing the p-adic unit associated to `z` modulo `z[3]` (and to zero if `z` is zero). To summarize, if $z \neq 0$, we have the equality:

$$\mathtt{z} = p^{\mathrm{valp(z)}} * (\mathtt{z}[4] + O(\mathtt{z}[3])), \quad \text{where} \quad \mathtt{z}[3] = O(p^{\mathrm{precp(z)}}).$$

4.5.8 Type `t_QUAD` (quadratic number). `z[1]` points to the canonical polynomial P defining the quadratic field (as output by `quadpoly`), `z[2]` to the "real part" and `z[3]` to the "imaginary part". The latter are of type `t_INT`, `t_FRAC`, `t_INTMOD`, or `t_PADIC` and are to be taken as the coefficients of `z` with respect to the canonical basis $(1, X)$ or $\mathbf{Q}[X]/(P(X))$, see Section **??**. Exact complex numbers may be implemented as quadratics, but `t_COMPLEX` is in general more versatile (`t_REAL` components are allowed) and more efficient.

Operations involving a `t_QUAD` and `t_COMPLEX` are implemented by converting the `t_QUAD` to a `t_REAL` (or `t_COMPLEX` with `t_REAL` components) to the accuracy of the `t_COMPLEX`. As a consequence, operations between `t_QUAD` and *exact* `t_COMPLEX`s are not allowed.

4.5.9 Type `t_POLMOD` (polmod). as for `t_INTMOD`s, `z[1]` points to the modulus, and `z[2]` to a polynomial representing the class of `z`. Both must be of type `t_POL` in the same variable, satisfying the inequality $\deg \mathtt{z}[2] < \deg \mathtt{z}[1]$. However, `z[2]` is allowed to be a simplification of such a polynomial, e.g. a scalar. This is tricky considering the hierarchical structure of the variables; in particular, a polynomial in variable of *lesser* priority (see Section **??**) than the modulus variable is valid, since it is considered as the constant term of a polynomial of degree 0 in the correct variable. On the other hand a variable of *greater* priority is not acceptable; see Section **??** for the problems which may arise.

4.5.10 Type `t_POL` (polynomial). this type has a second codeword. It contains a "*sign*": 0 if the polynomial is equal to 0, and 1 if not (see however the important remark below) and a *variable number* (e.g. 0 for x, 1 for y, etc...).

These data can be handled with the following macros: **signe** and **setsigne** as for `t_INT` and `t_REAL`,

`long varn(GEN z)` returns the variable number of the object `z`,

`void setvarn(GEN z, long v)` sets the variable number of `z` to `v`.

The variable numbers encode the relative priorities of variables as discussed in Section **??**. We will give more details in Section 4.6. Note also the function `long gvar(GEN z)` which tries to return a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE`, which has lower priority than any other variable number.

The components `z[2]`, `z[3]`,...`z[lg(z)-1]` point to the coefficients of the polynomial *in ascending order*, with `z[2]` being the constant term and so on.

For a `t_POL` of non-zero sign, `degpol`, `leading_term`, `constant_term`, return its degree, and a pointer to the leading, resp. constant, coefficient with respect to the main variable. Note that no copy is made on the PARI stack so the returned value is not safe for a basic `gerepile` call. Applied to any other type than `t_POL`, the result is unspecified. Those three functions are still defined when the sign is 0, see Section 5.2.7 and Section 9.4.

`long degree(GEN x)` returns the degree of `x` with respect to its main variable even when `x` is not a polynomial (a rational function for instance). By convention, the degree of a zero polynomial is -1.

Important remark. The leading coefficient of a `t_POL` may be equal to zero:

- it is not allowed to be an exact rational 0, such as `gen_0`;
- an exact non-rational 0, like `Mod(0,2)`, is possible for constant polynomials, i.e. of length 3 and no other coefficient: this carries information about the base ring for the polynomial;
- an inexact 0, like `0.E-38` or `O(3^5)`, is always possible. Inexact zeroes do not correspond to an actual 0, but to a very small coefficient according to some metric; we keep them to give information on how much cancellation occurred in previous computations.

A polynomial disobeying any of these rules is an invalid *unnormalized* object. We advise *not* to use low-level constructions to build a `t_POL` coefficient by coefficient, such as

```
GEN T = cgetg(4, t_POL);
T[1] = evalvarn(0);
gel(T, 2) = x;
gel(T, 3) = y;
```

But if you do and it is not clear whether the result will be normalized, call

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. This functions sets `signe` (to 0 or 1) properly.

Caveat. A consequence of the remark above is that zero polynomials are characterized by the fact that their sign is 0. It is in general incorrect to check whether `lg(x)` is 2 or `degpol(x)` < 0, although both tests are valid when the coefficient types are under control: for instance, when they are guaranteed to be `t_INT`s or `t_FRAC`s. The same remark applies to `t_SER`s.

4.5.11 Type `t_SER` (power series). This type also has a second codeword, which encodes a "*sign*", i.e. 0 if the power series is 0, and 1 if not, a *variable number* as for polynomials, and an *exponent*. This information can be handled with the following functions: **signe**, **setsigne**, **varn**, **setvarn** as for polynomials, and **valp**, **setvalp** for the exponent as for p-adic numbers. Beware: do *not* use **expo** and **setexpo** on power series.

The coefficients `z[2]`, `z[3]`,...`z[lg(z)-1]` point to the coefficients of `z` in ascending order. As for polynomials (see remark there), the sign of a `t_SER` is 0 if and only all its coefficients are equal to 0. (The leading coefficient cannot be an integer 0.)

Note that the exponent of a power series can be negative, i.e. we are then dealing with a Laurent series (with a finite number of negative terms).

4.5.12 Type `t_RFRAC` (rational function). `z[1]` points to the numerator n, and `z[2]` on the denominator d. The denominator must be of type `t_POL`, with variable of higher priority than the numerator. The numerator n is not an exact 0 and $(n, d) = 1$ (see `gred_rfac2`).

4.5.13 Type `t_QFR` (indefinite binary quadratic form). `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form and are of type `t_INT`. `z[4]` is Shanks's distance function, and must be of type `t_REAL`.

4.5.14 Type `t_QFI` (definite binary quadratic form). `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form. All three are of type `t_INT`.

4.5.15 Type `t_VEC` and `t_COL` (vector). `z[1]`, `z[2]`,...`z[lg(z)-1]` point to the components of the vector.

4.5.16 Type `t_MAT` (matrix). `z[1]`, `z[2]`,...`z[lg(z)-1]` point to the column vectors of `z`, i.e. they must be of type `t_COL` and of the same length.

4.5.17 Type `t_VECSMALL` (vector of small integers). `z[1]`, `z[2]`,...`z[lg(z)-1]` are ordinary signed long integers. This type is used instead of a `t_VEC` of `t_INT`s for efficiency reasons, for instance to implement efficiently permutations, polynomial arithmetic and linear algebra over small finite fields, etc.

4.5.18 Type `t_STR` (character string).

`char * GSTR(z)` (= `(z+1)`) points to the first character of the (`NULL`-terminated) string.

4.5.19 Type `t_ERROR` (error context). This type holds error messages, as well as details about the error, as returned by the exception handling system. The second codeword `z[1]` contains the error type (an `int`, as passed to `pari_err`). The subsequent words `z[2]`,...`z[lg(z)-1]` are `GEN`s containing additional data, depending on the error type.

4.5.20 Type `t_CLOSURE` (closure). This type holds GP functions and closures, in compiled form. The internal detail of this type is subject to change each time the GP language evolves. Hence we do not describe it here and refer to the Developer's Guide. However functions to create or to evaluate `t_CLOSURE`s are documented in Section 11.1.

`long closure_arity(GEN C)` returns the arity of the `t_CLOSURE`.

4.5.21 Type `t_LIST` (list). this type was introduced for specific `gp` use and is rather inefficient compared to a straightforward linked list implementation (it requires more memory, as well as many unnecessary copies). Hence we do not describe it here and refer to the Developer's Guide.

Implementation note. For the types including an exponent (or a valuation), we actually store a biased non-negative exponent (bit-ORing the biased exponent to the codeword), obtained by adding a constant to the true exponent: either `HIGHEXPOBIT` (for `t_REAL`) or `HIGHVALPBIT` (for `t_PADIC` and `t_SER`). Of course, this is encapsulated by the exponent/valuation-handling macros and needs not concern the library user.

4.6 PARI variables.

4.6.1 Multivariate objects.

We now consider variables and formal computations, and give the technical details corresponding to the general discussion in Section **??**. As we have seen in Section 4.5, the codewords for types `t_POL` and `t_SER` encode a "variable number". This is an integer, ranging from 0 to `MAXVARN`. Relative priorities may be ascertained using

`int varncmp(long v, long w)`

which is > 0, $= 0$, < 0 whenever v has lower, resp. same, resp. higher priority than w.

The way an object is considered in formal computations depends entirely on its "principal variable number" which is given by the function

`long gvar(GEN z)`

which returns a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE` which has lower priority than any valid variable number. The variable number of a recursive type which is not a polynomial or power series is the variable number with highest priority among its components. But for polynomials and power series only the "outermost" number counts (we directly access `varn`(x) in the codewords): the representation is not symmetrical at all.

Under `gp`, one needs not worry too much since the interpreter defines the variables as it sees them* and do the right thing with the polynomials produced (however, have a look at the remark in Section **??**).

But in library mode, they are tricky objects if you intend to build polynomials yourself (and not just let PARI functions produce them, which is less efficient). For instance, it does not make

* The first time a given identifier is read by the GP parser a new variable is created, and it is assigned a strictly lower priority than any variable in use at this point. On startup, before any user input has taken place, 'x' is defined in this way and has initially maximal priority (and variable number 0).

sense to have a variable number occur in the components of a polynomial whose main variable has a lower priority, even though PARI cannot prevent you from doing it; see Section **??** for a discussion of possible problems in a similar situation.

4.6.2 Creating variables. A basic difficulty is to "create" a variable. Some initializations are needed before you can use a given integer v as a variable number.

Initially, this is done for 0 (the variable `x` under `gp`), and `MAXVARN`, which is there to address the need for a "temporary" new variable in library mode and cannot be input under `gp`. No documented library function can create from scratch an object involving `MAXVARN` (of course, if the operands originally involve `MAXVARN`, the function abides). We call the latter type a "temporary variable". The regular variables meant to be used in regular objects, are called "user variables".

4.6.2.1 User variables. When the program starts, `x` is the only user variable (number 0). To define new ones, use

`long fetch_user_var(char *s)`: inspects the user variable whose name is the string pointed to by `s`, creating it if needed, and returns its variable number.

```
long v = fetch_user_var("y");
GEN gy = pol_x(v);
```

The function raises an exception if the name is already in use for an `installed` or built-in function, or an alias.

Caveat. You can use `gp_read_str` (see Section 4.7.1) to execute a GP command and create GP variables on the fly as needed:

```
GEN gy = gp_read_str("'y"); /* returns pol_x(v), for some v */
long v = varn(gy);
```

But please note the quote `'y` in the above. Using `gp_read_str("y")` might work, but is dangerous, especially when programming functions to be used under `gp`. The latter reads the value of `y`, as *currently* known by the `gp` interpreter, possibly creating it in the process. But if `y` has been modified by previous `gp` commands (e.g. `y = 1`), then the value of `gy` is not what you expected it to be and corresponds instead to the current value of the `gp` variable (e.g. `gen_1`).

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v. Returns `NULL` if that variable number is unknown to the interpreter, e.g. it is a user variable. Note that this may not be the same as `pol_x(v)` if assignments have been performed in the interpreter.

4.6.2.2 Temporary variables. `MAXVARN` is available, but is better left to PARI internal functions (some of which do not check that `MAXVARN` is free for them to use, which can be considered a bug). You can create more temporary variables using

`long fetch_var()`

This returns a variable number which is guaranteed to be unused by the library at the time you get it and as long as you do not delete it (we will see how to do that shortly). This has *higher* priority than any temporary variable produced so far (`MAXVARN` is assumed to be the first such). After the statement `v = fetch_var()`, you can use `pol_1(v)` and `pol_x(v)`. The variables created in this way have no identifier assigned to them though, and are printed as `#<number>`, except for `MAXVARN` which is printed as `#`. You can assign a name to a temporary variable, after creating it, by calling the function

`void name_var(long n, char *s)`

after which the output machinery will use the name `s` to represent the variable number `n`. The GP parser will *not* recognize it by that name, however, and calling this on a variable known to `gp` raises an error. Temporary variables are meant to be used as free variables, and you should never assign values or functions to them as you would do with variables under `gp`. For that, you need a user variable.

All objects created by `fetch_var` are on the heap and not on the stack, thus they are not subject to standard garbage collecting (they are not destroyed by a `gerepile` or `avma = ltop` statement). When you do not need a variable number anymore, you can delete it using

`long delete_var()`

which deletes the *latest* temporary variable created and returns the variable number of the previous one (or simply returns 0 if you try, in vain, to delete `MAXVARN`). Of course you should make sure that the deleted variable does not appear anywhere in the objects you use later on. Here is an example:

```
long first = fetch_var();
long n1 = fetch_var();
long n2 = fetch_var(); /* prepare three variables for internal use */
...
/* delete all variables before leaving */
do { num = delete_var(); } while (num && num <= first);
```

The (dangerous) statement

```
while (delete_var()) /* empty */;
```

removes all temporary variables in use, except `MAXVARN` which cannot be deleted.

4.7 Input and output.

Two important aspects have not yet been explained which are specific to library mode: input and output of PARI objects.

4.7.1 Input.

For input, PARI provides a powerful high level function which enables you to input your objects as if you were under `gp`. In fact, it *is* essentially the GP syntactical parser, hence you can use it not only for input but for (most) computations that you can do under `gp`. It has the following syntax:

`GEN gp_read_str(const char *s)`

Note that `gp`'s metacommands are not recognized.

Note. The obsolete form

`GEN readseq(char *t)`

still exists for backward compatibility (assumes filtered input, without spaces or comments). Don't use it.

To read a `GEN` from a file, you can use the simpler interface

`GEN gp_read_stream(FILE *file)`

which reads a character string of arbitrary length from the stream `file` (up to the first complete expression sequence), applies `gp_read_str` to it, and returns the resulting `GEN`. This way, you do not have to worry about allocating buffers to hold the string. To interactively input an expression, use `gp_read_stream(stdin)`.

Finally, you can read in a whole file, as in GP's `read` statement

`GEN gp_read_file(char *name)`

As usual, the return value is that of the last non-empty expression evaluated. There is one technical exception: if `name` is a *binary* file (from `writebin`) containing more than one object, a `t_VEC` containing them all is returned. This is because binary objects bypass the parser, hence reading them has no useful side effect.

4.7.2 Output to screen or file, output to string.

General output functions return nothing but print a character string as a side effect. Low level routines are available to write on PARI output stream `pari_outfile` (`stdout` by default):

`void pari_putc(char c)`: write character `c` to the output stream.

`void pari_puts(char *s)`: write `s` to the output stream.

`void pari_flush()`: flush output stream; most streams are buffered by default, this command makes sure that all characters output so are actually written.

`void pari_printf(const char *fmt, ...)`: the most versatile such function. `fmt` is a character string similar to the one `printf` uses. In there, `%` characters have a special meaning, and describe how to print the remaining operands. In addition to the standard format types (see the GP function `printf`), you can use the *length modifier* `P` (for PARI of course!) to specify that an argument is a `GEN`. For instance, the following are valid conversions for a `GEN` argument

`%Ps`	*convert to* `char*` *(will print an arbitrary* `GEN`*)*
`%P.10s`	*convert to* `char*`*, truncated to 10 chars*
`%P.2f`	*convert to floating point format with 2 decimals*
`%P4d`	*convert to integer, field width at least 4*

```
pari_printf("x[%d] = %Ps is not invertible!\n", i, gel(x,i));
```

Here `i` is an `int`, `x` a `GEN` which is not a leaf (presumably a vector, or a polynomial) and this would insert the value of its i-th `GEN` component: `gel(x,i)`.

Simple but useful variants to `pari_printf` are

`void output(GEN x)` prints `x` in raw format, followed by a newline and a buffer flush. This is more or less equivalent to

```
pari_printf("%Ps\n", x);
```

```
pari_flush();
```

`void outmat(GEN x)` as above except if x is a `t_MAT`, in which case a multi-line display is used to display the matrix. This is prettier for small dimensions, but quickly becomes unreadable and cannot be pasted and reused for input. If all entries of x are small integers, you may use the recursive features of `%Pd` and obtain the same (or better) effect with

```
pari_printf("%Pd\n", x);
pari_flush();
```

A variant like `"%5Pd"` would improve alignment by imposing 5 chars for each coefficient. Similarly if all entries are to be converted to floats, a format like `"%5.1Pf"` could be useful.

These functions write on (PARI's idea of) standard output, and must be used if you want your functions to interact nicely with `gp`. In most programs, this is not a concern and it is more flexible to write to an explicit `FILE*`, or to recover a character string:

`void pari_fprintf(FILE *file, const char *fmt, ...)` writes the remaining arguments to stream `file` according to the format specification `fmt`.

`char* pari_sprintf(const char *fmt, ...)` produces a string from the remaining arguments, according to the PARI format `fmt` (see `printf`). This is the `libpari` equivalent of `Strprintf`, and returns a `malloc`'ed string, which must be freed by the caller. Note that contrary to the analogous `sprintf` in the `libc` you do not provide a buffer (leading to all kinds of buffer overflow concerns); the function provided is actually closer to the GNU extension `asprintf`, although the latter has a different interface.

Simple variants of `pari_sprintf` convert a `GEN` to a `malloc`'ed ASCII string, which you must still `free` after use:

`char* GENtostr(GEN x)`, using the current default output format (`prettymat` by default).

`char* GENtoTeXstr(GEN x)`, suitable for inclusion in a TEX file.

Note that we have `va_list` analogs of the functions of `printf` type seen so far:

```
void pari_vprintf(const char *fmt, va_list ap)
```

```
void pari_vfprintf(FILE *file, const char *fmt, va_list ap)
```

```
char* pari_vsprintf(const char *fmt, va_list ap)
```

4.7.3 Errors.

If you want your functions to issue error messages, you can use the general error handling routine `pari_err`. The basic syntax is

```
pari_err(e_MISC, "error message");
```

This prints the corresponding error message and exit the program (in library mode; go back to the `gp` prompt otherwise). You can also use it in the more versatile guise

```
pari_err(e_MISC, format, ...);
```

where `format` describes the format to use to write the remaining operands, as in the `pari_printf` function. For instance:

```
pari_err(e_MISC, "x[%d] = %Ps is not invertible!", i, gel(x,i));
```

The simple syntax seen above is just a special case with a constant format and no remaining arguments. The general syntax is

`void pari_err(numerr,...)`

where `numerr` is a codeword which specifies the error class and what to do with the remaining arguments and what message to print. For instance, if x is a `GEN` with internal type `t_STR`, say, `pari_err(e_TYPE,"extgcd", x)` prints the message:

```
***   incorrect type in extgcd (t_STR),
```

See Section 10.4 for details. In the libpari code itself, the general-purpose `e_MISC` is used sparingly: it is so flexible that the corresponding error contexts (`t_ERROR`) become hard to use reliably. Other more rigid error types are generally more useful: for instance the error context associated to the `e_TYPE` exception above is precisely documented and contains `"extgcd"` and x (not only its type) as readily available components.

4.7.4 Warnings.

To issue a warning, use

`void pari_warn(warnerr,...)` In that case, of course, we do *not* abort the computation, just print the requested message and go on. The basic example is

```
pari_warn(warner, "Strategy 1 failed. Trying strategy 2")
```

which is the exact equivalent of `pari_err(e_MISC,...)` except that you certainly do not want to stop the program at this point, just inform the user that something important has occurred; in particular, this output would be suitably highlighted under `gp`, whereas a simple `printf` would not.

The valid *warning* keywords are `warner` (general), `warnprec` (increasing precision), `warnmem` (garbage collecting) and `warnfile` (error in file operation), used as follows:

```
pari_warn(warnprec, "bnfinit", newprec);
pari_warn(warnmem,  "bnfinit");
pari_warn(warnfile, "close", "afile");  /* error when closing "afile" */
```

4.7.5 Debugging output.

For debugging output, you can use the standard output functions, `output` and `pari_printf` mainly. Corresponding to the `gp` metacommand `\x`, you can also output the hexadecimal tree associated to an object:

`void dbgGEN(GEN x, long nb = -1)`, displays the recursive structure of `x`. If `nb` $= -1$, the full structure is printed, otherwise the leaves (non-recursive components) are truncated to `nb` words.

The function `output` is vital under debuggers, since none of them knows how to print PARI objects by default. Seasoned PARI developers add the following `gdb` macro to their `.gdbinit`:

```
define i
  call output((GEN)$arg0)
end
```

Typing `i x` at a breakpoint in `gdb` then prints the value of the `GEN x` (provided the optimizer has not put it into a register, but it is rarely a good idea to debug optimized code).

The global variables **DEBUGLEVEL** and **DEBUGMEM** (corresponding to the default **debug** and **debugmem**, see Section **??**) are used throughout the PARI code to govern the amount of diagnostic and debugging output, depending on their values. You can use them to debug your own functions, especially if you `install` the latter under `gp` (see Section **??**).

`void dbg_pari_heap(void)` print debugging statements about the PARI stack, heap, and number of variables used. Corresponds to `\s` under gp.

4.7.6 Timers and timing output.

To handle timings in a reentrant way, PARI defines a dedicated data type, `pari_timer`, together with the following methods:

`void timer_start(pari_timer *T)` start (or reset) a timer.

`long timer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Resets the timer as a side effect.

`long timer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer.

`long timer_printf(pari_timer *T, char *format,...)` This diagnostics function is equivalent to the following code

```
err_printf("Time ")
... prints remaining arguments according to format ...
err_printf(": %ld", timer_delay(T));
```

Resets the timer as a side effect.

They are used as follows:

```
pari_timer T;
timer_start(&T); /* initialize timer */
...
printf("Total time: %ldms\n", timer_delay(&T));
```

or

```
pari_timer T;
timer_start(&T);
for (i = 1; i < 10; i++) {
  ...
  timer_printf(&T, "for i = %ld (L[i] = %Ps)", i, gel(L,i));
}
```

The following functions provided the same functionality, in a non-reentrant way, and are now deprecated.

`long timer(void)`

`long timer2(void)`

`void msgtimer(const char *format, ...)`

The following function implements `gp`'s timer and should not be used in libpari programs: `long gettime(void)` equivalent to `timer_delay`(T) associated to a private timer T.

4.8 Iterators, Numerical integration, Sums, Products.

4.8.1 Iterators. Since it is easier to program directly simple loops in library mode, some GP iterators are mainly useful for GP programming. Here are the others:

• `fordiv` is a trivial iteration over a list produced by `divisors`.

• `forell` and `forsubgroup` are currently not implemented as an iterator but as a procedure with callbacks.

`void forell(void *E, long fun(void*, GEN), GEN a, GEN b)` goes through the same curves as `forell(ell,a,b,)`, calling `fun(E, ell)` for each curve `ell`, stopping if `fun` returns a non-zero value.

`void forsubgroup(void *E, long fun(void*, GEN), GEN G, GEN B)` goes through the same subgroups as `forsubgroup(H = G, B,)`, calling `fun(E, H)` for each subgroup H, stopping if `fun` returns a non-zero value.

• `forprime`, for which we refer you to the next subsection.

• `forcomposite`, we provide an iterator over composite integers:

`int forcomposite(forcomposite_t *T, GEN a, GEN b)` initialize an iterator T over composite integers in $[a, b]$; over composites $\geq a$ if $b =$ `NULL`. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise.

`GEN forcomposite_next(forcomposite_t *T)` returns the next composite in the range, assuming that T was initialized by `forcomposite_init`.

• `forvec`, for which we provide a convenient iterator. To initialize the analog of `forvec(X = v, ..., flag)`, call

`int forvec_init(forvec_t *T, GEN v, long flag)` initialize an iterator T over the vectors generated by `forvec(X =` v`,..., flag)`. This returns 0 if this vector list is empty, and 1 otherwise.

`GEN forvec_next(forvec_t *T)` returns the next element in the `forvec` sequence, or `NULL` if we are done. The return value must be used immediately or copied since the next call to the iterator destroys it: the relevant vector is updated in place. The iterator works hard to not use up PARI stack, and is more efficient when all lower bounds in the initialization vector v are integers. In that case, the cost is linear in the number of tuples enumerated, and you can expect to run over more than 10^9 tuples per minute. If speed is critical and all integers involved would fit in C `long`s, write a simple direct backtracking algorithm yourself.

• `forpart` is a variant of `forvec` which iterates over partitions. See the documentation of the `forpart` GP function for details. This function is available as a loop with callbacks:

`void forpart(void *data, long (*call)(void*,GEN), long k, GEN a, GEN n)`

It is also available as an iterator:

`void forpart_init(forpart_t *T, long k, GEN a, GEN n)` initializes an iterator over the partitions of k, with length restricted by n, and components restricted by a, either of which can be set to `NULL` to run without restriction.

`GEN forpart_next(forpart_t *T)` returns the next partition, or `NULL` when all partitions have been exhausted.

`GEN forpart_prev(forpart_t *T)` returns the previous partition, or `NULL` when all partitions have been exhausted.

You may *not* mix calls to `forpart_next` and `forpart_prev`: the first one called determines the ordering used to iterate over the partitions; you can not go back since the `forpart_t` structure is used in incompatible ways.

4.8.2 Iterating over primes.

The library provides a high-level iterator, which stores its (private) data in a `struct forprime_t` and runs over arbitrary ranges of primes, without ever overflowing.

The iterator has two flavors, one providing the successive primes as `ulong`s, the other as `GEN`. They are initialized as follows, where we expect to run over primes $\geq a$ and $\leq b$:

`int forprime_init(forprime_t *T, GEN a, GEN b)` for the `GEN` variant, where $b =$ `NULL` means $+\infty$.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)` for the `ulong` variant, where $b =$ `ULONG_MAX` means we will run through all primes representable in a `ulong` type.

Both variant return 1 on success, and 0 if the iterator would run over an empty interval (if $a > b$, for instance). They allocate the `forprime_t` data structure on the PARI stack.

The successive primes are then obtained using

`GEN forprime_next(forprime_t *T)`, returns `NULL` if no more primes are available in the interval.

`ulong u_forprime_next(forprime_t *T)`, returns 0 if no more primes are available in the interval.

These two functions leave alone the PARI stack, and write their state information in the preallocated `forprime_t` struct. The typical usage is thus:

```
forprime_t T;
GEN p;
pari_sp av = avma, av2;

forprime_init(&T, gen_2, stoi(1000));
av2 = avma;
while ( (p = forprime_next(&T)) )
{
  ...
  if ( prime_is_OK(p) ) break;
  avma = av2; /* delete garbage accumulated in this iteration */
}
avma = av; /* delete all */
```

Of course, the final `avma = av` could be replaced by a `gerepile` call. Beware that swapping the `av2 = avma` and `forprime_init` call would be incorrect: the first `avma = av2` would delete the `forprime_t` structure!

4.8.3 Numerical analysis.

Numerical routines code a function (to be integrated, summed, zeroed, etc.) with two parameters named

```
void *E;
GEN (*eval)(void*, GEN)
```

The second is meant to contain all auxiliary data needed by your function. The first is such that `eval(x, E)` returns your function evaluated at `x`. For instance, one may code the family of functions $f_t : x \to (x+t)^2$ via

```
GEN fun(void *t, GEN x) { return gsqr(gadd(x, (GEN)t)); }
```

One can then integrate f_1 between a and b with the call

```
intnum((void*)stoi(1), &fun, a, b, NULL, prec);
```

Since you can set `E` to a pointer to any `struct` (typecast to `void*`) the above mechanism handles arbitrary functions. For simple functions without extra parameters, you may set `E = NULL` and ignore that argument in your function definition.

4.9 Catching exceptions.

4.9.1 Basic use.

PARI provides a mechanism to trap exceptions generated via `pari_err` using the `pari_CATCH` construction. The basic usage is as follows

```
pari_CATCH(err_code) {
   recovery branch
}
pari_TRY {
   main branch
}
pari_ENDCATCH
```

This fragment executes the main branch, then the recovery branch *if* exception `err_code` is thrown, e.g. `e_TYPE`. See Section 10.4 for the description of all error classes. The special error code `CATCH_ALL` is available to catch all errors.

One can replace the `pari_TRY` keyword by `pari_RETRY`, in which case once the recovery branch is run, we run the main branch again, still catching the same exceptions.

Restrictions.

• Such constructs can be nested without adverse effect, the innermost handler catching the exception.

• It is *valid* to leave either branch using `pari_err`.

• It is *invalid* to use C flow control instructions (`break`, `continue`, `return`) to directly leave either branch without seeing the `pari_ENDCATCH` keyword. This would leave an invalid structure in the exception handler stack, and the next exception would crash.

• In order to leave using `break`, `continue` or `return`, one must precede the keyword by a call to

`void pari_CATCH_reset()` disable the current handler, allowing to leave without adverse effect.

4.9.2 Advanced use.

In the recovery branch, the exception context can be examined via the following helper routines:

`GEN pari_err_last()` returns the exception context, as a `t_ERROR`. The exception E returned by `pari_err_last` can be rethrown, using

```
pari_err(0, E);
```

`long err_get_num(GEN E)` returns the error symbolic name. E.g `e_TYPE`.

`GEN err_get_compo(GEN E, long i)` error i-th component, as documented in Section 10.4.

For instance

```
pari_CATCH(CATCH_ALL) { /* catch everything */
   GEN x, E = pari_err_last();
   long code = err_get_num(E);
   if (code != e_INV) pari_err(0, E); /* unexpected error, rethrow */
   x = err_get_compo(E, 2);
   /* e_INV has two components, 1: function name 2: non-invertible x */
   if (typ(x) != t_INTMOD) pari_err(0, E); /* unexpected type, rethrow */
   pari_CATCH_reset();
   return x; /* leave ! */
    ...
} pari_TRY {
   main branch
}
pari_ENDCATCH
```

4.10 A complete program.

Now that the preliminaries are out of the way, the best way to learn how to use the library mode is to study a detailed example. We want to write a program which computes the gcd of two integers, together with the Bezout coefficients. We shall use the standard quadratic algorithm which is not optimal but is not too far from the one used in the PARI function **bezout**.

Let x, y two integers and initially $\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To apply the ordinary Euclidean algorithm to the right hand side, multiply the system from the left by $\begin{pmatrix} 0 & 1 \\ 1 & -q \end{pmatrix}$, with $q = \texttt{floor}(x/y)$. Iterate until $y = 0$ in the right hand side, then the first line of the system reads

$$s_x x + s_y y = \gcd(x, y).$$

In practice, there is no need to update s_y and t_y since $\gcd(x, y)$ and s_x are enough to recover s_y. The following program is now straightforward. A couple of new functions appear in there, whose description can be found in the technical reference manual in Chapter 5, but whose meaning should be clear from their name and the context.

This program can be found in `examples/extgcd.c` together with a proper `Makefile`. You may ignore the first comment

```
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
```

which instruments the program so that `gp2c-run extgcd.c` can import the `extgcd()` routine into an instance of the `gp` interpreter (under the name `gcdex`). See the `gp2c` manual for details.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;
  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviiexact( subii(a, mulii(A,ux)), B );
  gerepileall(av, 3, &a, U, V); return a;
}
int
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

For simplicity, the inner loop does not include any garbage collection, hence memory use is quadratic in the size of the inputs instead of linear. Here is a better version of that loop:

```
  pari_sp av = avma, lim = stack_lim(av,1);
  ...
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
    if (low_stack(lim, stack_lim(av,1)))
      gerepileall(av, 4, &a, &b, &ux, &vx);
```

```
}
```

Chapter 5:
Technical Reference Guide: the basics

In the following chapters, we describe all public low-level functions of the PARI library. These include specialized functions for handling all the PARI types. Simple higher level functions, such as arithmetic or transcendental functions, are described in Chapter 3 of the GP user's manual; we will eventually see more general or flexible versions in the chapters to come. A general introduction to the major concepts of PARI programming can be found in Chapter 4, which you should really read first.

We shall now study specialized functions, more efficient than the library wrappers, but sloppier on argument checking and damage control; besides speed, their main advantage is to give finer control about the inner workings of generic routines, offering more options to the programmer.

Important advice. Generic routines eventually call lower level functions. Optimize your algorithms first, not overhead and conversion costs between PARI routines. For generic operations, use generic routines first; do not waste time looking for the most specialized one available unless you identify a genuine bottleneck, or you need some special behavior the generic routine does not offer. The PARI source code is part of the documentation; look for inspiration there.

The type `long` denotes a `BITS_IN_LONG`-bit signed long integer (32 or 64 bits). The type `ulong` is defined as `unsigned long`. The word *stack* always refer to the PARI stack, allocated through an initial `pari_init` call. Refer to Chapters 1–2 and 4 for general background.

We shall often refer to the notion of *shallow* function, which means that some components of the result may point to components of the input, which is more efficient than a *deep* copy (full recursive copy of the object tree). Such outputs are not suitable for `gerepileupto` and particular care must be taken when garbage collecting objects which have been input to shallow functions: corresponding outputs also become invalid and should no longer be accessed.

A function is *not stack clean* if it leaves intermediate data on the stack besides its output, for efficiency reasons.

5.1 Initializing the library.

The following functions enable you to start using the PARI functions in a program, and cleanup without exiting the whole program.

5.1.1 General purpose.

`void pari_init(size_t size, ulong maxprime)` initialize the library, with a stack of `size` bytes and a prime table up to the maximum of `maxprime` and 2^{16}. Unless otherwise mentioned, no PARI function will function properly before such an initialization.

`void pari_close(void)` stop using the library (assuming it was initialized with `pari_init`) and frees all allocated objects.

5.1.2 Technical functions.

`void pari_init_opts(size_t size, ulong maxprime, ulong opts)` as `pari_init`, more flexible. `opts` is a mask of flags among the following:

`INIT_JMPm`: install PARI error handler. When an exception is raised, the program is terminated with `exit(1)`.

`INIT_SIGm`: install PARI signal handler.

`INIT_DFTm`: initialize the `GP_DATA` environment structure. This one *must* be enabled once. If you close pari, then restart it, you need not reinitialize `GP_DATA`; if you do not, then old values are restored.

`INIT_noPRIMEm`: do not compute the prime table (ignore the `maxprime` argument). The user *must* call `initprimetable` later.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_init` to initialize the multi-thread engine. If this flag is set, `pari_mt_init()` will need to be called manually. See `examples/pari-mt.c` for an example.

`void pari_close_opts(ulong init_opts)` as `pari_close`, for a library initialized with a mask of options using `pari_init_opts`. `opts` is a mask of flags among

`INIT_SIGm`: restore `SIG_DFL` default action for signals tampered with by PARI signal handler.

`INIT_DFTm`: frees the `GP_DATA` environment structure.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_close` to close the multi-thread engine.

`void pari_sig_init(void (*f)(int))` install the signal handler `f` (see `signal(2)`): the signals `SIGBUS`, `SIGFPE`, `SIGINT`, `SIGBREAK`, `SIGPIPE` and `SIGSEGV` are concerned.

`void pari_stackcheck_init(void *stackbase)` controls the system stack exhaustion checking code in the GP interpreter. This should be used when the system stack base address change or when the address seen by `pari_init` is too far from the base address. If `stackbase` is `NULL`, disable the check, else set the base address to `stackbase`. It is normally used this way

```
int thread_start (...)
{
  long first_item_on_the_stack;
  ...
  pari_stackcheck_init(&first_item_on_the_stack);
}
```

`int pari_daemon(void)` fork a PARI daemon, detaching from the main process group. The function returns 1 in the parent, and 0 in the forked son.

5.1.3 Notions specific to the GP interpreter.

An `entree` is the generic object associated to an identifier (a name) in GP's interpreter, be it a built-in or user function, or a variable. For a function, it has at least the following fields:

`char *name`: the name under which the interpreter knows us.

`void *value`: a pointer to the C function to call.

`long menu`: an integer from 1 to 11 (to which group of function help do we belong).

`char *code`: the prototype code.

`char *help`: the help text for the function.

A routine in GP is described to the analyzer by an `entree` structure. Built-in PARI routines are grouped in *modules*, which are arrays of `entree` structs, the last of which satisfy `name = NULL` (sentinel).

There are currently five modules in PARI/GP: general functions (`functions_basic`, known to `libpari`), gp-specific functions (`functions_gp`), gp-specific highlevel functions (`functions_highlevel`), and two modules of obsolete functions. The function `pari_init` initializes the interpreter and declares all symbols in `functions_basic`. You may declare further functions on a case by case basis or as a whole module using

`void pari_add_function(entree *ep)` adds a single routine to the table of symbols in the interpreter. It assumes `pari_init` has been called.

`void pari_add_module(entree *mod)` adds all the routines in module `mod` to the table of symbols in the interpreter. It assumes `pari_init` has been called.

For instance, gp implements a number of private routines, which it adds to the default set via the calls

```
pari_add_module(functions_gp);
pari_add_module(functions_highlevel);
```

`void pari_add_oldmodule(entree *mod)` adds all the routines in module `mod` to the table of symbols in the interpreter when running in "PARI 1.xx compatible" mode (see `default(compatible)`) . It assumes that `pari_init` has been called.

A GP `default` is likewise associated to a helper routine, that is run when the value is consulted, or changed by `default0` or `setdefault`. Such routines are grouped into modules: `functions_default` containing all defaults that make sense in libpari context, `functions_gp_rl_default` containing defaults that are gp-specific and do not make sense unless we use `libreadline`, and `functions_gp_default` containing all other gp-specific defaults.

`void pari_add_defaults_module(entree *mod)` adds all the defaults in module `mod` to the interpreter. It assumes that `pari_init` has been called. From this point on, all defaults in module `mod` are known to `setdefault` and friends.

5.1.4 Public callbacks.

The gp calculator associates elaborate functions (for instance the break loop handler) to the following callbacks, and so can you:

`void (*cb_pari_ask_confirm)(const char *s)` initialized to `NULL`. Called with argument s whenever PARI wants confirmation for action s, for instance in `secure` mode.

`extern int (*cb_pari_handle_exception)(long)` initialized to `NULL`. If not `NULL`, this routine is called with argument -1 on `SIGINT`, and argument `err` on error `err`. If it returns a non-zero value, the error or signal handler returns, in effect further ignoring the error or signal, otherwise it raises a fatal error.

`void (*cb_pari_sigint)(void)`. Function called when we receive `SIGINT`. By default, raises

```
pari_err(e_MISC, "user interrupt");
```

`extern void (*cb_pari_err_recover)(long)` initialized to `NULL`. If not `NULL`, this routine is called just before PARI cleans up from an error. It is not required to return. The error number is passed as argument, unless the PARI stack has been destroyed (`allocatemem`), in which case -1 is passed.

`extern void (*cb_pari_err_recover)(long)` initialized to `pari_exit()`. This callback must not return. It is called after PARI has cleaned-up from an error. The error number is passed as argument, unless the PARI stack has been destroyed, in which case it is called with argument -1.

`int (*cb_pari_whatnow)(PariOUT *out, const char *s, int flag)` initialized to `NULL`. If not `NULL`, must check whether s existed in older versions of `pari` (the gp callback checks against `pari-1.39.15`). All output must be done via `out` methods.

- *flag* $= 0$: should print verbosely the answer, including help text if available.

- *flag* $= 1$: must return 0 if the function did not change, and a non-0 result otherwise. May print a help message.

5.1.5 Configuration variables.

`pari_library_path`: If set, It should be a path to the libpari library. It is used by the function `gpinstall` to locate the PARI library when searching for symbols. This should only be useful on Windows.

Utility function.

`void pari_ask_confirm(const char *s)` raise an error if the callback `cb_pari_ask_confirm` is `NULL`. Otherwise calls

```
cb_pari_ask_confirm(s);
```

5.1.6 Saving and restoring the GP context.

`void gp_context_save(struct gp_context* rec)` save the current GP context.

`void gp_context_restore(struct gp_context* rec)` restore a GP context. The new context must be an ancestor of the current context.

5.1.7 GP history.

These functions allow to control the GP history (the `%` operator).

`void pari_add_hist(GEN x, long t)` adds `x` as the last history entry; t is the time we used to compute it.

`GEN pari_get_hist(long p)`, if $p > 0$ returns entry of index p (i.e. `%p`), else returns entry of index $n + p$ where n is the index of the last entry (used for `%`, `%‘`, `%‘‘`, etc.).

`long pari_get_histtime(long p)` as `pari_get_hist`, returning the time used to compute the history entry, instead of the entry itself.

`ulong pari_nb_hist(void)` return the index of the last entry.

5.2 Handling `GEN`s.

Almost all these functions are either macros or inlined. Unless mentioned otherwise, they do not evaluate their arguments twice. Most of them are specific to a set of types, although no consistency checks are made: e.g. one may access the `sign` of a `t_PADIC`, but the result is meaningless.

5.2.1 Allocation.

`GEN cgetg(long l, long t)` allocates (the root of) a `GEN` of type t and length l. Sets $z[0]$.

`GEN cgeti(long l)` allocates a `t_INT` of length l (including the 2 codewords). Sets $z[0]$ only.

`GEN cgetr(long l)` allocates a `t_REAL` of length l (including the 2 codewords). Sets $z[0]$ only.

`GEN cgetc(long prec)` allocates a `t_COMPLEX` whose real and imaginary parts are `t_REAL`s of length `prec`.

`GEN cgetg_copy(GEN x, long *lx)` fast version of `cgetg`: allocate a `GEN` with the same type and length as x, setting `*lx` to `lg(x)` as a side-effect. (Only sets the first codeword.) This is a little faster than `cgetg` since we may reuse the bitmask in $x[0]$ instead of recomputing it, and we do not need to check that the length does not overflow the possibilities of the implementation (since an object with that length already exists). Note that `cgetg` with arguments known at compile time, as in

```
cgetg(3, t_INTMOD)
```

will be even faster since the compiler will directly perform all computations and checks.

`GEN vectrunc_init(long l)` perform `cgetg(l,t_VEC)`, then set the length to 1 and return the result. This is used to implement vectors whose final length is easily bounded at creation time, that we intend to fill gradually using:

`void vectrunc_append(GEN x, GEN y)` assuming x was allocated using `vectrunc_init`, appends y as the last element of x, which grows in the process. The function is shallow: we append y, not a copy; it is equivalent to

```
long lx = lg(x); gel(x,lx) = y; setlg(x, lx+1);
```

Beware that the maximal size of x (the l argument to `vectrunc_init`) is unknown, hence unchecked, and stack corruption will occur if we append more than $l - 1$ elements to x. Use the safer (but slower) `shallowconcat` when l is not easy to bound in advance.

An other possibility is simply to allocate using `cgetg(l, t)` then fill the components as they become available: this time the downside is that we do not obtain a correct `GEN` until the vector is complete. Almost no PARI function will be able to operate on it.

`void vectrunc_append_batch(GEN x, GEN y)` successively apply

```
vectrunc_append(x, gel(y, i))
```

for all elements of the vector y.

`GEN vecsmalltrunc_init(long l)`

`void vecsmalltrunc_append(GEN x, long t)` analog to the above for a `t_VECSMALL` container.

5.2.2 Length conversions.

These routines convert a non-negative length to different units. Their behavior is undefined at negative integers.

`long ndec2nlong(long x)` converts a number of decimal digits to a number of words. Returns $1 + \texttt{floor}(x \times \texttt{BITS\_IN\_LONG} \log_2 10)$.

`long ndec2prec(long x)` converts a number of decimal digits to a number of codewords. This is equal to $2 +$ `ndec2nlong(x)`.

`long prec2ndec(long x)` converts a number of of codewords to a number of decimal digits.

`long nbits2nlong(long x)` converts a number of bits to a number of words. Returns the smallest word count containing x bits, i.e `ceil`$(x/$`BITS_IN_LONG`$)$.

`long nbits2prec(long x)` converts a number of bits to a number of codewords. This is equal to $2 +$ `nbits2nlong(x)`.

`long nchar2nlong(long x)` converts a number of bytes to number of words. Returns the smallest word count containing x bytes, i.e `ceil`$(x/$`sizeof(long)`$)$.

`long bit_accuracy(long x)` converts a `t_REAL` length into a number of significant bits. Returns $(x-2)$`BITS_IN_LONG`.

`double bit_accuracy_mul(long x, double y)` returns $(x-2)$`BITS_IN_LONG` $\times y$.

5.2.3 Read type-dependent information.

`long typ(GEN x)` returns the type number of `x`. The header files included through `pari.h` define symbolic constants for the `GEN` types: `t_INT` etc. Never use their actual numerical values. E.g to determine whether `x` is a `t_INT`, simply check

```
if (typ(x) == t_INT) { }
```

The types are internally ordered and this simplifies the implementation of commutative binary operations (e.g addition, gcd). Avoid using the ordering directly, as it may change in the future; use type grouping functions instead (Section 5.2.6).

`const char* type_name(long t)` given a type number `t` this routine returns a string containing its symbolic name. E.g `type_name(t_INT)` returns `"t_INT"`. The return value is read-only.

`long lg(GEN x)` returns the length of `x` in `BITS_IN_LONG`-bit words.

`long lgefint(GEN x)` returns the effective length of the `t_INT` `x` in `BITS_IN_LONG`-bit words.

`long signe(GEN x)` returns the sign (-1, 0 or 1) of `x`. Can be used for `t_INT`, `t_REAL`, `t_POL` and `t_SER` (for the last two types, only 0 or 1 are possible).

`long gsigne(GEN x)` returns the sign of a real number x, valid for `t_INT`, `t_REAL` as `signe`, but also for `t_FRAC`. Raise a type error if `typ(x)` is not among those three.

`long expi(GEN x)` returns the binary exponent of the real number equal to the `t_INT` `x`. This is a special case of `gexpo`.

`long expo(GEN x)` returns the binary exponent of the `t_REAL` `x`.

`long mpexpo(GEN x)` returns the binary exponent of the `t_INT` or `t_REAL` `x`.

`long gexpo(GEN x)` same as `expo`, but also valid when `x` is not a `t_REAL` (returns the largest exponent found among the components of `x`). When `x` is an exact 0, this returns `-HIGHEXPOBIT`, which is lower than any valid exponent.

`long valp(GEN x)` returns the p-adic valuation (for a `t_PADIC`) or X-adic valuation (for a `t_SER`, taken with respect to the main variable) of `x`.

`long precp(GEN x)` returns the precision of the `t_PADIC` `x`.

`long varn(GEN x)` returns the variable number of the `t_POL` or `t_SER` `x` (between 0 and `MAXVARN`).

`long gvar(GEN x)` returns the main variable number when any variable at all occurs in the composite object `x` (the smallest variable number which occurs), and `NO_VARIABLE` otherwise.

`long gvar2(GEN x)` returns the variable number for the ring over which x is defined, e.g. if $x \in \mathbf{Z}[a][b]$ return (the variable number for) a. Return `NO_VARIABLE` if x has no variable or is not defined over a polynomial ring.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL` `x` with respect to its main variable, *if* its leading coefficient is non-zero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1.

`long lgpol(GEN x)` is equal to `degpol(x) + 1`. Used to loop over the coefficients of a `t_POL` in the following situation:

```
GEN xd = x + 2;
long i, l = lgpol(x);
for (i = 0; i < l; i++) foo( xd[i] ).
```

`long precision(GEN x)` If `x` is of type `t_REAL`, returns the precision of `x`, namely the length of `x` in `BITS_IN_LONG`-bit words if `x` is not zero, and a reasonable quantity obtained from the exponent of `x` if `x` is numerically equal to zero. If `x` is of type `t_COMPLEX`, returns the minimum of the precisions of the real and imaginary part. Otherwise, returns 0 (which stands for infinite precision).

`long lgcols(GEN x)` is equal to `lg(gel(x,1))`. This is the length of the columns of a `t_MAT` with at least one column.

`long nbrows(GEN x)` is equal to `lg(gel(x,1))-1`. This is the number of rows of a `t_MAT` with at least one column.

`long gprecision(GEN x)` as `precision` for scalars. Returns the lowest precision encountered among the components otherwise.

`long sizedigit(GEN x)` returns 0 if `x` is exactly 0. Otherwise, returns **`gexpo(x)`** multiplied by $\log_{10}(2)$. This gives a crude estimate for the maximal number of decimal digits of the components of `x`.

5.2.4 Eval type-dependent information. These routines convert type-dependent information to bitmask to fill the codewords of `GEN` objects (see Section 4.5). E.g for a `t_REAL` `z`:

```
z[1] = evalsigne(-1) | evalexpo(2)
```

Compatible components of a codeword for a given type can be OR-ed as above.

`ulong evaltyp(long x)` convert type `x` to bitmask (first codeword of all `GEN`s)

`long evallg(long x)` convert length `x` to bitmask (first codeword of all `GEN`s). Raise overflow error if `x` is so large that the corresponding length cannot be represented

`long _evallg(long x)` as `evallg` *without* the overflow check.

`ulong evalvarn(long x)` convert variable number `x` to bitmask (second codeword of `t_POL` and `t_SER`)

`long evalsigne(long x)` convert sign `x` (in $-1, 0, 1$) to bitmask (second codeword of `t_INT`, `t_REAL`, `t_POL`, `t_SER`)

`long evalprecp(long x)` convert p-adic (X-adic) precision `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding precision cannot be represented.

`long _evalprecp(long x)` same as `evalprecp` *without* the overflow check.

`long evalvalp(long x)` convert p-adic (X-adic) valuation `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding valuation cannot be represented.

`long _evalvalp(long x)` same as `evalvalp` *without* the overflow check.

`long evalexpo(long x)` convert exponent `x` to bitmask (second codeword of `t_REAL`). Raise overflow error if `x` is so large that the corresponding exponent cannot be represented

`long _evalexpo(long x)` same as `evalexpo` *without* the overflow check.

`long evallgefint(long x)` convert effective length `x` to bitmask (second codeword `t_INT`). This should be less or equal than the length of the `t_INT`, hence there is no overflow check for the effective length.

5.2.5 Set type-dependent information. Use these functions and macros with extreme care since usually the corresponding information is set otherwise, and the components and further codeword fields (which are left unchanged) may not be compatible with the new information.

`void settyp(GEN x, long s)` sets the type number of `x` to `s`.

`void setlg(GEN x, long s)` sets the length of `x` to `s`. This is an efficient way of truncating vectors, matrices or polynomials.

`void setlgefint(GEN x, long s)` sets the effective length of the `t_INT` `x` to `s`. The number `s` must be less than or equal to the length of `x`.

`void setsigne(GEN x, long s)` sets the sign of `x` to `s`. If `x` is a `t_INT` or `t_REAL`, `s` must be equal to -1, 0 or 1, and if `x` is a `t_POL` or `t_SER`, `s` must be equal to 0 or 1. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void togglesign(GEN x)` sets the sign s of `x` to $-s$, in place.

`void togglesign_safe(GEN *x)` sets the s sign of `*x` to $-s$, in place, unless `*x` is one of the integer universal constants in which case replace `*x` by its negation (e.g. replace `gen_1` by `gen_m1`).

`void setabssign(GEN x)` sets the sign s of `x` to $|s|$, in place.

`void affectsign(GEN x, GEN y)` shortcut for `setsigne(y, signe(x))`. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void affectsign_safe(GEN x, GEN *y)` sets the sign of `*y` to that of `x`, in place, unless `*y` is one of the integer universal constants in which case replace `*y` by its negation if needed (e.g. replace `gen_1` by `gen_m1` if `x` is negative). No other sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void normalize_frac(GEN z)` assuming z is of the form `mkfrac(a,b)` with $b \neq 0$, make sure that $b > 0$ by changing the sign of a in place if needed (use `togglesign`).

`void setexpo(GEN x, long s)` sets the binary exponent of the `t_REAL` `x` to `s`. The value `s` must be a 24-bit signed number.

`void setvalp(GEN x, long s)` sets the p-adic or X-adic valuation of `x` to `s`, if `x` is a `t_PADIC` or a `t_SER`, respectively.

`void setprecp(GEN x, long s)` sets the p-adic precision of the `t_PADIC` `x` to `s`.

`void setvarn(GEN x, long s)` sets the variable number of the `t_POL` or `t_SER` `x` to `s` (where $0 \leq$ `s` $\leq$ `MAXVARN`).

5.2.6 Type groups. In the following functions, `t` denotes the type of a `GEN`. They used to be implemented as macros, which could evaluate their argument twice; *no longer*: it is not inefficient to write

```
is_intreal_t(typ(x))
```

`int is_recursive_t(long t)` `true` iff `t` is a recursive type (the non-recursive types are `t_INT`, `t_REAL`, `t_STR`, `t_VECSMALL`). Somewhat contrary to intuition, `t_LIST` is also non-recursive, ; see the Developer's guide for details.

`int is_intreal_t(long t)` `true` iff `t` is `t_INT` or `t_REAL`.

`int is_rational_t(long t)` `true` iff `t` is `t_INT` or `t_FRAC`.

`int is_vec_t(long t)` `true` iff `t` is `t_VEC` or `t_COL`.

`int is_matvec_t(long t)` `true` iff `t` is `t_MAT`, `t_VEC` or `t_COL`.

`int is_scalar_t(long t)` `true` iff `t` is a scalar, i.e a `t_INT`, a `t_REAL`, a `t_INTMOD`, a `t_FRAC`, a `t_COMPLEX`, a `t_PADIC`, a `t_QUAD`, or a `t_POLMOD`.

`int is_extscalar_t(long t)` `true` iff `t` is a scalar (see `is_scalar_t`) or `t` is `t_POL`.

`int is_const_t(long t)` `true` iff `t` is a scalar which is not `t_POLMOD`.

`int is_noncalc_t(long t)` true if generic operations (`gadd`, `gmul`) do not make sense for t: corresponds to types `t_LIST`, `t_STR`, `t_VECSMALL`, `t_CLOSURE`

5.2.7 Accessors and components. The first two functions return `GEN` components as copies on the stack:

`GEN compo(GEN x, long n)` creates a copy of the `n`-th true component (i.e. not counting the codewords) of the object `x`.

`GEN truecoeff(GEN x, long n)` creates a copy of the coefficient of degree `n` of `x` if `x` is a scalar, `t_POL` or `t_SER`, and otherwise of the `n`-th component of `x`.

On the contrary, the following routines return the address of a `GEN` component. No copy is made on the stack:

`GEN constant_term(GEN x)` returns the address of the constant coefficient of `t_POL x`. By convention, a 0 polynomial (whose `sign` is 0) has `gen_0` constant term.

`GEN leading_term(GEN x)` returns the address of the leading coefficient of `t_POL x`, i.e. the coefficient of largest index stored in the array representing x. This may be an inexact 0. By convention, return `gen_0` if the coefficient array is empty.

`GEN gel(GEN x, long i)` returns the address of the `x[i]` entry of `x`. (`el` stands for element.)

`GEN gcoeff(GEN x, long i, long j)` returns the address of the `x[i,j]` entry of `t_MAT x`, i.e. the coefficient at row `i` and column `j`.

`GEN gmael(GEN x, long i, long j)` returns the address of the `x[i][j]` entry of `x`. (`mael` stands for multidimensional array element.)

`GEN gmael2(GEN A, long x1, long x2)` is an alias for `gmael`. Similar macros `gmael3`, `gmael4`, `gmael5` are available.

5.3 Global numerical constants.

These are defined in the various public PARI headers.

5.3.1 Constants related to word size.

`long BITS_IN_LONG` $= 2^{\texttt{TWOPOTBITS\_IN\_LONG}}$: number of bits in a `long` (32 or 64).

`long BITS_IN_HALFULONG`: `BITS_IN_LONG` divided by 2.

`long LONG_MAX`: the largest positive `long`.

`ulong ULONG_MAX`: the largest `ulong`.

`long DEFAULTPREC`: the length (`lg`) of a `t_REAL` with 64 bits of accuracy

`long MEDDEFAULTPREC`: the length (`lg`) of a `t_REAL` with 128 bits of accuracy

`long BIGDEFAULTPREC`: the length (`lg`) of a `t_REAL` with 192 bits of accuracy

`ulong HIGHBIT`: the largest power of 2 fitting in an `ulong`.

`ulong LOWMASK`: bitmask yielding the least significant bits.

`ulong HIGHMASK`: bitmask yielding the most significant bits.

The last two are used to implement the following convenience macros, returning half the bits of their operand:

`ulong LOWWORD(ulong a)` returns least significant bits.

`ulong HIGHWORD(ulong a)` returns most significant bits.

Finally

`long divsBIL(long n)` returns the Euclidean quotient of n by `BITS_IN_LONG` (with non-negative remainder).

`long remsBIL(n)` returns the (non-negative) Euclidean remainder of n by `BITS_IN_LONG`

`long dvmdsBIL(long n, long *r)`

`ulong dvmduBIL(ulong n, ulong *r)` sets r to `remsBIL(n)` and returns `divsBIL(n)`.

5.3.2 Masks used to implement the `GEN` type.

These constants are used by higher level macros, like `typ` or `lg`:

`EXPOnumBITS`, `LGnumBITS`, `SIGNnumBITS`, `TYPnumBITS`, `VALPnumBITS`, `VARNnumBITS`: number of bits used to encode `expo`, `lg`, `signe`, `typ`, `valp`, `varn`.

`PRECPSHIFT`, `SIGNSHIFT`, `TYPSHIFT`, `VARNSHIFT`: shifts used to recover or encode `precp`, `varn`, `typ`, `signe`

`CLONEBIT`, `EXPOBITS`, `LGBITS`, `PRECPBITS`, `SIGNBITS`, `TYPBITS`, `VALPBITS`, `VARNBITS`: bitmasks used to extract `isclone`, `expo`, `lg`, `precp`, `signe`, `typ`, `valp`, `varn` from `GEN` codewords.

`MAXVARN`: the largest possible variable number.

`NO_VARIABLE`: sentinel returned by `gvar(x)` when `x` does not contain any polynomial; has a lower priority than any valid variable number.

`HIGHEXPOBIT`: a power of 2, one more that the largest possible exponent for a `t_REAL`.

`HIGHVALPBIT`: a power of 2, one more that the largest possible valuation for a `t_PADIC` or a `t_SER`.

5.3.3 $\log 2$, π.

These are `double` approximations to useful constants:

`LOG2`: $\log 2$.

`LOG10_2`: $\log 2/\log 10$.

`LOG2_10`: $\log 10/\log 2$.

`PI`: π.

5.4 Iterating over small primes, low-level interface.

One of the methods used by the high-level prime iterator (see Section 4.8.2), is a precomputed table. Its direct use is deprecated, but documented here.

After `pari_init(size, maxprime)`, a "prime table" is initialized with the successive *differences* of primes up to (possibly just a little beyond) `maxprime`. The prime table occupies roughly `maxprime`/ log(`maxprime`) bytes in memory, so be sensible when choosing `maxprime`; it is 500000 by default under `gp` and there is no real benefit in choosing a much larger value: the high-level iterator provide *fast* access to primes up to the *square* of `maxprime`. In any case, the implementation requires that `maxprime` $< 2^{\texttt{BITS\_IN\_LONG}} - 2048$, whatever memory is available.

PARI currently guarantees that the first 6547 primes, up to and including 65557, are present in the table, even if you set `maxprime` to zero. in the `pari_init` call.

Some convenience functions:

`ulong maxprime()` the largest prime computable using our prime table.

`void maxprime_check(ulong B)` raise an error if `maxprime()` is $< B$.

After the following initializations (the names p and ptr are arbitrary of course)

```
byteptr ptr = diffptr;
ulong p = 0;
```

calling the macro `NEXT_PRIME_VIADIFF_CHECK`(p, ptr) repeatedly will assign the successive prime numbers to p. Overrunning the prime table boundary will raise the error `e_MAXPRIME`, which will just print the error message:

```
*** not enough precomputed primes, need primelimit ~c
```

(for some numerical value c), then abort the computation. The alternative macro `NEXT_PRIME_VIADIFF` operates in the same way, but will omit that check, and is slightly faster. It should be used in the following way:

```
byteptr ptr = diffptr;
ulong p = 0;

if (maxprime() < goal) pari_err_MAXPRIME(goal); /* not enough primes */
while (p <= goal) /* run through all primes up to goal */
{
  NEXT_PRIME_VIADIFF(p, ptr);
  ...
}
```

Here, we use the general error handling function `pari_err` (see Section 4.7.3), with the codeword `e_MAXPRIME`, raising the "not enough primes" error. This could be rewritten as

```
maxprime_check(goal);
while (p <= goal) /* run through all primes up to goal */
{
  NEXT_PRIME_VIADIFF(p, ptr);
  ...
}
```

`bytepr initprimes(ulong maxprime, long *L, ulong *lastp)` computes a (malloc'ed) "prime table", in fact a table of all prime differences for $p <$ `maxprime` (and possibly a little beyond). Set L to the table length (argument to `malloc`), and *lastp* to the last prime in the table.

`void initprimetable(ulong maxprime)` computes a prime table (of all prime differences for $p <$ `maxprime`) and assign it to the global variable `diffptr`. Don't change `diffptr` directly, call this function instead. This calls `initprimes` and updates internal data recording the table size.

`ulong init_primepointer_geq(ulong a, byteptr *pd)` returns the smallest prime $p \geq a$, and sets `*pd` to the proper offset of `diffptr` so that `NEXT_PRIME_VIADIFF(p, *pd)` correctly returns `unextprime(p + 1)`.

`ulong init_primepointer_gt(ulong a, byteptr *pd)` returns the smallest prime $p > a$.

`ulong init_primepointer_leq(ulong a, byteptr *pd)` returns the largest prime $p \leq a$.

`ulong init_primepointer_lt(ulong a, byteptr *pd)` returns the largest prime $p < a$.

5.5 Handling the PARI stack.

5.5.1 Allocating memory on the stack.

`GEN cgetg(long n, long t)` allocates memory on the stack for an object of length `n` and type `t`, and initializes its first codeword.

`GEN cgeti(long n)` allocates memory on the stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetr(long n)` allocates memory on the stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_REAL)`.

`GEN cgetc(long n)` allocates memory on the stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s of length `n`.

`GEN cgetp(GEN x)` creates space sufficient to hold the `t_PADIC` `x`, and sets the prime p and the p-adic precision to those of `x`, but does not copy (the p-adic unit or zero representative and the modulus of) `x`.

`GEN new_chunk(size_t n)` allocates a `GEN` with n components, *without* filling the required code words. This is the low-level constructor underlying `cgetg`, which calls `new_chunk` then sets the first code word. It works by simply returning the address `((GEN)avma) - n`, after checking that it is larger than `(GEN)bot`.

`char* stack_malloc(size_t n)` allocates memory on the stack for n chars (*not* n `GEN`s). This is faster than using `malloc`, and easier to use in most situations when temporary storage is needed. In particular there is no need to `free` individually all variables thus allocated: a simple `avma = oldavma` might be enough. On the other hand, beware that this is not permanent independent storage, but part of the stack.

`char* stack_calloc(size_t n)` as `stack_malloc`, setting the memory to zero.

Objects allocated through these last three functions cannot be `gerepile`'d, since they are not yet valid `GEN`s: their codewords must be filled first.

`GEN cgetalloc(long t, size_t l)`, same as `cgetg(t, l)`, except that the result is allocated using `pari_malloc` instead of the PARI stack. The resulting `GEN` is now impervious to garbage collecting routines, but should be freed using `pari_free`.

5.5.2 Stack-independent binary objects.

`GENbin* copy_bin(GEN x)` copies x into a malloc'ed structure suitable for stack-independent binary transmission or storage. The object obtained is architecture independent provided, `sizeof(long)` remains the same on all PARI instances involved, as well as the multiprecision kernel (either native or GMP).

`GENbin* copy_bin_canon(GEN x)` as `copy_bin`, ensuring furthermore that the binary object is independent of the multiprecision kernel. Slower than `copy_bin`.

`GEN bin_copy(GENbin *p)` assuming p was created by `copy_bin(x)` (not necessarily by the same PARI instance: transmission or external storage may be involved), restores x on the PARI stack.

The routine `bin_copy` transparently encapsulate the following functions:

`GEN GENbinbase(GENbin *p)` the `GEN` data actually stored in p. All addresses are stored as offsets with respect to a common reference point, so the resulting `GEN` is unusable unless it is a non-recursive type; private low-level routines must be called first to restore absolute addresses.

`void shiftaddress(GEN x, long dec)` converts relative addresses to absolute ones.

`void shiftaddress_canon(GEN x, long dec)` converts relative addresses to absolute ones, and converts leaves from a canonical form to the one specific to the multiprecision kernel in use. The `GENbin` type stores whether leaves are stored in canonical form, so `bin_copy` can call the right variant.

Objects containing closures are harder to e.g. copy and save to disk, since closures contain pointers to libpari functions that will not be valid in another gp instance: there is little chance for them to be loaded at the exact same address in memory. Such objects must be saved along with a linking table.

`GEN copybin_unlink(GEN C)` returns a linking table allowing to safely store and transmit `t_CLOSURE` objects in C. If C = `NULL` return a linking table corresponding to the content of all gp variables. C may then be dumped to disk in binary form, for instance.

`void bincopy_relink(GEN C, GEN V)` given a binary object C, as dumped by writebin and read back into a session, and a linking table V, restore all closures contained in C (function pointers are translated to their current value).

5.5.3 Garbage collection.

See Section 4.3 for a detailed explanation and many examples.

`void cgiv(GEN x)` frees object `x`, assuming it is the last created on the stack.

`GEN gerepile(pari_sp p, pari_sp q, GEN x)` general garbage collector for the stack.

`void gerepileall(pari_sp av, int n, ...)` cleans up the stack from `av` on (i.e from `avma` to `av`), preserving the `n` objects which follow in the argument list (of type `GEN*`). For instance, `gerepileall(av, 2, &x, &y)` preserves `x` and `y`.

`void gerepileallsp(pari_sp av, pari_sp ltop, int n, ...)` cleans up the stack between `av` and `ltop`, updating the `n` elements which follow `n` in the argument list (of type `GEN*`). Check that the elements of `g` have no component between `av` and `ltop`, and assumes that no garbage is present between `avma` and `ltop`. Analogous to (but faster than) `gerepileall` otherwise.

`GEN gerepilecopy(pari_sp av, GEN x)` cleans up the stack from `av` on, preserving the object `x`. Special case of `gerepileall` (case `n` = 1), except that the routine returns the preserved `GEN` instead of updating its address through a pointer.

`void gerepilemany(pari_sp av, GEN* g[], int n)` alternative interface to `gerepileall`. The preserved `GEN`s are the elements of the array `g` of length n: `g[0]`, `g[1]`, ..., `g[`n`-1]`. Obsolete: no more efficient than `gerepileall`, error-prone, and clumsy (need to declare an extra `GEN *g`).

`void gerepilemanysp(pari_sp av, pari_sp ltop, GEN* g[], int n)` alternative interface to `gerepileallsp`. Obsolete.

`void gerepilecoeffs(pari_sp av, GEN x, int n)` cleans up the stack from `av` on, preserving `x[0]`, ..., `x[n-1]` (which are `GEN`s).

`void gerepilecoeffssp(pari_sp av, pari_sp ltop, GEN x, int n)` cleans up the stack from `av` to `ltop`, preserving `x[0]`, ..., `x[n-1]` (which are `GEN`s). Same assumptions as in `gerepilemanysp`, of which this is a variant. For instance

```
z = cgetg(3, t_COMPLEX);
av = avma; garbage(); ltop = avma;
z[1] = fun1();
z[2] = fun2();
gerepilecoeffssp(av, ltop, z + 1, 2);
return z;
```

cleans up the garbage between `av` and `ltop`, and connects `z` and its two components. This is marginally more efficient than the standard

```
av = avma; garbage(); ltop = avma;
z = cgetg(3, t_COMPLEX);
z[1] = fun1();
z[2] = fun2(); return gerepile(av, ltop, z);
```

`GEN gerepileupto(pari_sp av, GEN q)` analogous to (but faster than) `gerepilecopy`. Assumes that `q` is connected and that its root was created before any component. If `q` is not on the stack, this is equivalent to `avma = av`; in particular, sentinels which are not even proper `GEN`s such as `q = NULL` are allowed.

`GEN gerepileuptoint(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a `t_INT`. The length and effective length of the resulting `t_INT` are equal.

`GEN gerepileuptoleaf(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a leaf, i.e a non-recursive type (`is_recursive_t(typ(q))` is non-zero). Contrary to `gerepileuptoint` and `gerepileupto`, `gerepileuptoleaf` leaves length and effective length of a `t_INT` unchanged.

5.5.4 Garbage collection: advanced use.

`void stackdummy(pari_sp av, pari_sp ltop)` inhibits the memory area between `av` *included* and `ltop` *excluded* with respect to `gerepile`, in order to avoid a call to `gerepile(av, ltop,...)`. The stack space is not reclaimed though.

More precisely, this routine assumes that `av` is recorded earlier than `ltop`, then marks the specified stack segment as a non-recursive type of the correct length. Thus gerepile will not inspect the zone, at most copy it. To be used in the following situation:

```
av0 = avma; z = cgetg(t_VEC, 3);
gel(z,1) = HUGE(); av = avma; garbage(); ltop = avma;
```

```
gel(z,2) = HUGE(); stackdummy(av, ltop);
```

Compared to the orthodox

```
gel(z,2) = gerepile(av, ltop, gel(z,2));
```

or even more wasteful

```
z = gerepilecopy(av0, z);
```

we temporarily lose (`av`$-$`ltop`) words but save a costly `gerepile`. In principle, a garbage collection higher up the call chain should reclaim this later anyway.

Without the `stackdummy`, if the [`av`, `ltop`] zone is arbitrary (not even valid `GEN`s as could happen after direct truncation via `setlg`), we would leave dangerous data in the middle of `z`, which would be a problem for a later

```
gerepile(..., ... , z);
```

And even if it were made of valid `GEN`s, inhibiting the area makes sure `gerepile` will not inspect their components, saving time.

Another natural use in low-level routines is to "shorten" an existing `GEN z` to its first `n` -1 components:

```
setlg(z, n);
stackdummy((pari_sp)(z + lg(z)), (pari_sp)(z + n));
```

or to its last `n` components:

```
long L = lg(z) - n, tz = typ(z);
stackdummy((pari_sp)(z + L), (pari_sp)z);
z += L; z[0] = evaltyp(tz) | evallg(L);
```

The first scenario (safe shortening an existing `GEN`) is in fact so common, that we provide a function for this:

`void fixlg(GEN z, long ly)` a safe variant of `setlg(z, ly)`. If `ly` is larger than `lg(z)` do nothing. Otherwise, shorten z in place, using `stackdummy` to avoid later `gerepile` problems.

`GEN gcopy_avma(GEN x, pari_sp *AVMA)` return a copy of x as from `gcopy`, except that we pretend that initially `avma` is `*AVMA`, and that `*AVMA` is updated accordingly (so that the total size of x is the difference between the two successive values of `*AVMA`). It is not necessary for `*AVMA` to initially point on the stack: `gclone` is implemented using this mechanism.

`GEN icopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a `t_INT` and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

5.5.5 Debugging the PARI stack.

`int chk_gerepileupto(GEN x)` returns 1 if `x` is suitable for `gerepileupto`, and 0 otherwise. In the latter case, print a warning explaining the problem.

`void dbg_gerepile(pari_sp ltop)` outputs the list of all objects on the stack between `avma` and `ltop`, i.e. the ones that would be inspected in a call to `gerepile(...,ltop,...)`.

`void dbg_gerepileupto(GEN q)` outputs the list of all objects on the stack that would be inspected in a call to `gerepileupto(...,q)`.

5.5.6 Copies.

`GEN gcopy(GEN x)` creates a new copy of x on the stack.

`GEN gcopy_lg(GEN x, long l)` creates a new copy of x on the stack, pretending that `lg(x)` is l, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `gcopy(x)`.

`int isonstack(GEN x)` `true` iff x belongs to the stack.

`void copyifstack(GEN x, GEN y)` sets `y = gcopy(x)` if x belongs to the stack, and `y = x` otherwise. This macro evaluates its arguments once, contrary to

```
y = isonstack(x)? gcopy(x): x;
```

`void icopyifstack(GEN x, GEN y)` as `copyifstack` assuming `x` is a `t_INT`.

5.5.7 Simplify.

`GEN simplify(GEN x)` you should not need that function in library mode. One rather uses:

`GEN simplify_shallow(GEN x)` shallow, faster, version of `simplify`.

5.6 The PARI heap.

5.6.1 Introduction.

It is implemented as a doubly-linked list of `malloc`'ed blocks of memory, equipped with reference counts. Each block has type `GEN` but need not be a valid `GEN`: it is a chunk of data preceded by a hidden header (meaning that we allocate x and return $x +$ `headersize`). A *clone*, created by `gclone`, is a block which is a valid `GEN` and whose *clone bit* is set.

5.6.2 Public interface.

`GEN newblock(size_t n)` allocates a block of n *words* (not bytes).

`void killblock(GEN x)` deletes the block x created by `newblock`. Fatal error if x not a block.

`GEN gclone(GEN x)` creates a new permanent copy of x on the heap (allocated using `newblock`). The *clone bit* of the result is set.

`GEN gcloneref(GEN x)` if x is not a clone, clone it and return the result; otherwise, increase the clone reference count and return x.

`void gunclone(GEN x)` deletes a clone. Deletion at first only decreases the reference count by 1. If the count remains positive, no further action is taken; if the count becomes zero, then the clone is actually deleted. In the current implementation, this is an alias for `killblock`, but it is cleaner to kill clones (valid `GEN`s) using this function, and other blocks using `killblock`.

`void gunclone_deep(GEN x)` is only useful in the context of the GP interpreter which may replace arbitrary components of container types (`t_VEC`, `t_COL`, `t_MAT`, `t_LIST`) by clones. If x is such a container, the function recursively deletes all clones among the components of x, then unclones x. Useless in library mode: simply use `gunclone`.

`void traverseheap(void(*f)(GEN, void *), void *data)` this applies `f(`x`, data)` to each object x on the PARI heap, most recent first. Mostly for debugging purposes.

`GEN getheap()` a simple wrapper around `traverseheap`. Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words.

`GEN cgetg_block(long x, long y)` as `cgetg(x,y)`, creating the return value as a `block`, not on the PARI stack.

`GEN cgetr_block(long prec)` as `cgetr(prec)`, creating the return value as a `block`, not on the PARI stack.

5.6.3 Implementation note. The hidden block header is manipulated using the following private functions:

`void* bl_base(GEN x)` returns the pointer that was actually allocated by `malloc` (can be freed).

`long bl_refc(GEN x)` the reference count of x: the number of pointers to this block. Decremented in `killblock`, incremented by the private function `void gclone_refc(GEN x)`; block is freed when the reference count reaches 0.

`long bl_num(GEN x)` the index of this block in the list of all blocks allocated so far (including freed blocks). Uniquely identifies a block until $2^{\texttt{BITS\_IN\_LONG}}$ blocks have been allocated and this wraps around.

`GEN bl_next(GEN x)` the block *after* x in the linked list of blocks (`NULL` if x is the last block allocated not yet killed).

`GEN bl_prev(GEN x)` the block allocated *before* x (never `NULL`).

We documented the last four routines as functions for clarity (and type checking) but they are actually macros yielding valid lvalues. It is allowed to write `bl_refc(x)++` for instance.

5.7 Handling user and temp variables.

Low-level implementation of user / temporary variables is liable to change. We describe it nevertheless for completeness. Currently variables are implemented by a single array of values divided in 3 zones: 0–`nvar` (user variables), `max_avail`–`MAXVARN` (temporary variables), and `nvar+1`–`max_avail-1` (pool of free variable numbers).

5.7.1 Low-level.

`void pari_var_init()`: a small part of `pari_init`. Resets variable counters `nvar` and `max_avail`, notwithstanding existing variables! In effect, this even deletes `x`. Don't use it.

`long pari_var_next()`: returns `nvar`, the number of the next user variable we can create.

`long pari_var_next_temp()` returns `max_avail`, the number of the next temp variable we can create.

`void pari_var_create(entree *ep)` low-level initialization of an `EpVAR`.

The obsolete function `long manage_var(long n, entree *ep)` is kept for backward compatibility only. Don't use it.

5.7.2 User variables.

`long fetch_user_var(char *s)` returns a user variable whose name is `s`, creating it is needed (and using an existing variable otherwise). Returns its variable number.

`entree* fetch_named_var(char *s)` as `fetch_user_var`, but returns an `entree*` suitable for inclusion in the interpreter hashlists of symbols, not a variable number. `fetch_user_var` is a trivial wrapper.

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v. Return `NULL` for a temporary variable.

`entree* is_entry(const char *s)` returns the `entree*` associated to an identifier `s` (variable or function), from the interpreter hashtables. Return `NULL` is the identifier is unknown.

5.7.3 Temporary variables.

`long fetch_var(void)` returns the number of a new temporary variable (decreasing `max_avail`).

`long delete_var(void)` delete latest temp variable created and return the number of previous one.

`void name_var(long n, char *s)` rename temporary variable number `n` to `s`; mostly useful for nicer printout. Error when trying to rename a user variable: use `fetch_named_var` to get a user variable of the right name in the first place.

5.8 Adding functions to PARI.

5.8.1 Nota Bene. As mentioned in the `COPYING` file, modified versions of the PARI package can be distributed under the conditions of the GNU General Public License. If you do modify PARI, however, it is certainly for a good reason, and we would like to know about it, so that everyone can benefit from your changes. There is then a good chance that your improvements are incorporated into the next release.

We classify changes to PARI into four rough classes, where changes of the first three types are almost certain to be accepted. The first type includes all improvements to the documentation, in a broad sense. This includes correcting typos or inaccuracies of course, but also items which are not really covered in this document, e.g. if you happen to write a tutorial, or pieces of code exemplifying fine points unduly omitted in the present manual.

The second type is to expand or modify the configuration routines and skeleton files (the `Configure` script and anything in the `config/` subdirectory) so that compilation is possible (or easier, or more efficient) on an operating system previously not catered for. This includes discovering and removing idiosyncrasies in the code that would hinder its portability.

The third type is to modify existing (mathematical) code, either to correct bugs, to add new functionality to existing functions, or to improve their efficiency.

Finally the last type is to add new functions to PARI. We explain here how to do this, so that in particular the new function can be called from `gp`.

5.8.2 Coding guidelines. Code your function in a file of its own, using as a guide other functions in the PARI sources. One important thing to remember is to clean the stack before exiting your main function, since otherwise successive calls to the function clutters the stack with unnecessary garbage, and stack overflow occurs sooner. Also, if it returns a `GEN` and you want it to be accessible to `gp`, you have to make sure this `GEN` is suitable for `gerepileupto` (see Section 4.3).

If error messages or warnings are to be generated in your function, use `pari_err` and `pari_warn` respectively. Recall that `pari_err` does not return but ends with a `longjmp` statement. As well, instead of explicit `printf` / `fprintf` statements, use the following encapsulated variants:

`void pari_putc(char c)`: write character `c` to the output stream.

`void pari_puts(char *s)`: write `s` to the output stream.

`void pari_printf(const char *fmt, ...)`: write following arguments to the output stream, according to the conversion specifications in format `fmt` (see `printf`).

`void err_printf(const char *fmt, ...)`: as `pari_printf`, writing to PARI's current error stream.

`void err_flush(void)` flush error stream.

Declare all public functions in an appropriate header file, if you want to access them from C. The other functions should be declared `static` in your file.

Your function is now ready to be used in library mode after compilation and creation of the library. If possible, compile it as a shared library (see the `Makefile` coming with the `extgcd` example in the distribution). It is however still inaccessible from `gp`.

5.8.3 GP prototypes, parser codes. A *GP prototype* is a character string describing all the GP parser needs to know about the function prototype. It contains a sequence of the following atoms:

• Return type: `GEN` by default (must be valid for `gerepileupto`), otherwise the following can appear as the *first* char of the code string:

`i`	return `int`
`l`	return `long`
`v`	return `void`
`m`	return a `GEN` which is not `gerepile`-safe.

The `m` code is used for member functions, to avoid unnecessary copies. A copy opcode is generated by the compiler if the result needs to be kept safe for later use.

• Mandatory arguments, appearing in the same order as the input arguments they describe:

`G`	`GEN`
`&`	`*GEN`
`L`	`long` (we implicitly typecast `int` to `long`)
`V`	loop variable
`n`	variable, expects a variable number (a `long`, not an `*entree`)
`W`	a `GEN` which is a lvalue to be modified in place (for `t_LIST`)
`r`	raw input (treated as a string without quotes). Quoted args are copied as strings Stops at first unquoted `')'` or `','`. Special chars can be quoted using `'\'` Example: `aa"b\n)"c` yields the string `"aab\n)c"`
`s`	expanded string. Example: `Pi"x"2` yields `"3.142x2"`

Unquoted components can be of any PARI type, converted to string following current output format

`I` closure whose value is ignored, as in `for` loops, to be processed by `void closure_evalvoid(GEN C)`

`E` closure whose value is used, as in `sum` loops, to be processed by `void closure_evalgen(GEN C)`

`J` implicit function of arity 1, as in `parsum` loops, to be processed by `void closure_callgen1(GEN C)`

A *closure* is a GP function in compiled (bytecode) form. It can be efficiently evaluated using the `closure_eval`*xxx* functions.

• Automatic arguments:

`f` Fake `*long`. C function requires a pointer but we do not use the resulting `long`

`p` real precision (default `realprecision`)

`P` series precision (default `seriesprecision`, global variable `precdl` for the library)

`C` lexical context (internal, for `eval`, see `localvars_read_str`)

• Syntax requirements, used by functions like `for`, `sum`, etc.:

`=` separator `=` required at this point (between two arguments)

• Optional arguments and default values:

`E*` any number of expressions, possibly 0 (see `E`)

`s*` any number of strings, possibly 0 (see `s`)

`D`*xxx* argument can be omitted and has a default value

The `E*` code reads all remaining arguments in closure context and passes them as a single `t_VEC`. The `s*` code reads all remaining arguments in *string context* (see Section **??**), and passes the list of strings as a single `t_VEC`. The automatic concatenation rules in string context are implemented so that adjacent strings are read as different arguments, as if they had been comma-separated. For instance, if the remaining argument sequence is: `"xx" 1, "yy"`, the `s*` atom sends `[a, b, c]`, where *a*, *b*, *c* are `GEN`s of type `t_STR` (content `"xx"`), `t_INT` (equal to 1) and `t_STR` (content `"yy"`).

The format to indicate a default value (atom starts with a `D`) is "`D`*value*`,`*type*`,`", where *type* is the code for any mandatory atom (previous group), *value* is any valid GP expression which is converted according to *type*, and the ending comma is mandatory. For instance `D0,L,` stands for "this optional argument is converted to a `long`, and is `0` by default". So if the user-given argument reads `1 + 3` at this point, `4L` is sent to the function; and `0L` if the argument is omitted. The following special notations are available:

`DG` optional `GEN`, send `NULL` if argument omitted.

`D&` optional `*GEN`, send `NULL` if argument omitted. The argument must be prefixed by `&`.

`Dr` optional raw string, send `NULL` if argument omitted.

`Ds` optional `char *`, send `NULL` if argument omitted.

`DV` optional `*entree`, send `NULL` if argument omitted.

`DI`, `DE` optional closure, send `NULL` if argument omitted.

`Dn` optional variable number, -1 if omitted.

Hardcoded limit. C functions using more than 20 arguments are not supported. Use vectors if you really need that many parameters.

When the function is called under `gp`, the prototype is scanned and each time an atom corresponding to a mandatory argument is met, a user-given argument is read (`gp` outputs an error message it the argument was missing). Each time an optional atom is met, a default value is inserted if the user omits the argument. The "automatic" atoms fill in the argument list transparently, supplying the current value of the corresponding variable (or a dummy pointer).

For instance, here is how you would code the following prototypes, which do not involve default values:

```
GEN f(GEN x, GEN y, long prec)   ----> "GGp"
void f(GEN x, GEN y, long prec)  ----> "vGGp"
void f(GEN x, long y, long prec) ----> "vGLp"
long f(GEN x)                    ----> "lG"
int f(long x)                    ----> "iL"
```

If you want more examples, `gp` gives you easy access to the parser codes associated to all GP functions: just type `\h` *function*. You can then compare with the C prototypes as they stand in `paridecl.h`.

Remark. If you need to implement complicated control statements (probably for some improved summation functions), you need to know how the parser implements closures and lexicals and how the evaluator lets you deal with them, in particular the `push_lex` and `pop_lex` functions. Check their descriptions and adapt the source code in `language/sumiter.c` and `language/intnum.c`.

5.8.4 Integration with `gp` as a shared module.

In this section we assume that your Operating System is supported by `install`. You have written a function in C following the guidelines is Section 5.8.2; in case the function returns a `GEN`, it must satisfy `gerepileupto` assumptions (see Section 4.3).

You then succeeded in building it as part of a shared library and want to finally tell `gp` about your function. First, find a name for it. It does not have to match the one used in library mode, but consistency is nice. It has to be a valid GP identifier, i.e. use only alphabetic characters, digits and the underscore character (_), the first character being alphabetic.

Then figure out the correct parser code corresponding to the function prototype (as explained in Section 5.8.3) and write a GP script like the following:

```
install(libname, code, gpname, library)
addhelp(gpname, "some help text")
```

(see Section **??** and **??**). The addhelp part is not mandatory, but very useful if you want others to use your module. `libname` is how the function is named in the library, usually the same name as one visible from C.

Read that file from your `gp` session, for instance from your preferences file (Section **??**), and that's it. You can now use the new function *gpname* under `gp`, and we would very much like to hear about it!

Example. A complete description could look like this:

```
{
  install(bnfinit0, "GD0,L,DGp", ClassGroupInit, "libpari.so");
  addhelp(ClassGroupInit, "ClassGroupInit(P,{flag=0},{data=[]}):
    compute the necessary data for ...");
}
```

which means we have a function `ClassGroupInit` under `gp`, which calls the library function `bnfinit0` . The function has one mandatory argument, and possibly two more (two `'D'` in the code), plus the current real precision. More precisely, the first argument is a `GEN`, the second one is converted to a `long` using `itos` (`0` is passed if it is omitted), and the third one is also a `GEN`, but we pass `NULL` if no argument was supplied by the user. This matches the C prototype (from `paridecl.h`):

```
GEN bnfinit0(GEN P, long flag, GEN data, long prec)
```

This function is in fact coded in `basemath/buch2.c`, and is in this case completely identical to the GP function `bnfinit` but `gp` does not need to know about this, only that it can be found somewhere in the shared library `libpari.so`.

Important note. You see in this example that it is the function's responsibility to correctly interpret its operands: `data = NULL` is interpreted *by the function* as an empty vector. Note that since `NULL` is never a valid `GEN` pointer, this trick always enables you to distinguish between a default value and actual input: the user could explicitly supply an empty vector!

5.8.5 Library interface for `install`.

There is a corresponding library interface for this `install` functionality, letting you expand the GP parser/evaluator available in the library with new functions from your C source code. Functions such as `gp_read_str` may then evaluate a GP expression sequence involving calls to these new function!

```
entree * install(void *f, const char *gpname, const char *code)
```

where `f` is the (address of the) function (cast to `void*`), `gpname` is the name by which you want to access your function from within your GP expressions, and `code` is as above.

5.8.6 Integration by patching `gp`.

If `install` is not available, and installing Linux or a BSD operating system is not an option (why?), you have to hardcode your function in the `gp` binary. Here is what needs to be done:

• Fetch the complete sources of the PARI distribution.

• Drop the function source code module in an appropriate directory (a priori `src/modules`), and declare all public functions in `src/headers/paridecl.h`.

• Choose a help section and add a file `src/functions/`*section*`/`*gpname* containing the following, keeping the notation above:

`Function:` *gpname*
`Section:` *section*
`C-Name:` *libname*
`Prototype:` *code*
`Help:` *some help text*

(If the help text does not fit on a single line, continuation lines must start by a whitespace character.) Two GP2C-related fields (`Description` and `Wrapper`) are also available to improve the code GP2C generates when compiling scripts involving your function. See the GP2C documentation for details.

• Launch `Configure`, which should pick up your C files and build an appropriate `Makefile`. At this point you can recompile `gp`, which will first rebuild the functions database.

Example. We reuse the `ClassGroupInit` / `bnfinit0` from the preceding section. Since the C source code is already part of PARI, we only need to add a file

```
functions/number_fields/ClassGroupInit
```

containing the following:

```
Function: ClassGroupInit
Section: number_fields
C-Name: bnfinit0
Prototype: GD0,L,DGp
Help: ClassGroupInit(P,{flag=0},{tech=[]}): this routine does ...
```

and recompile `gp`.

5.9 Globals related to PARI configuration.

5.9.1 PARI version numbers.

`paricfg_version_code` encodes in a single `long`, the Major and minor version numbers as well as the patchlevel.

`long PARI_VERSION(long M, long m, long p)` produces the version code associated to release $M.m.p$. Each code identifies a unique PARI release, and corresponds to the natural total order on the set of releases (bigger code number means more recent release).

`PARI_VERSION_SHIFT` is the number of bits used to store each of the integers M, m, p in the version code.

`paricfg_vcsversion` is a version string related to the revision control system used to handle your sources, if any. For instance `git-`*commit hash* if compiled from a git repository.

The two character strings `paricfg_version` and `paricfg_buildinfo`, correspond to the first two lines printed by `gp` just before the Copyright message. The character string `paricfg_compiledate` is the date of compilation which appears on the next line. The character string `paricfg_mt_engine` is the name of the threading engine on the next line.

`GEN pari_version()` returns the version number as a PARI object, a `t_VEC` with three `t_INT` and one `t_STR` components.

5.9.2 Miscellaneous.

`paricfg_datadir`: character string. The location of PARI's `datadir`.

Chapter 6:
Arithmetic kernel: Level 0 and 1

6.1 Level 0 kernel (operations on ulongs).

6.1.1 Micro-kernel. The Level 0 kernel simulates basic operations of the 68020 processor on which PARI was originally implemented. They need "global" `ulong` variables `overflow` (which will contain only 0 or 1) and `hiremainder` to function properly. A routine using one of these lowest-level functions where the description mentions either `hiremainder` or `overflow` must declare the corresponding

```
LOCAL_HIREMAINDER;  /* provides 'hiremainder' */
LOCAL_OVERFLOW;     /* provides 'overflow' */
```

in a declaration block. Variables `hiremainder` and `overflow` then become available in the enclosing block. For instance a loop over the powers of an `ulong p` protected from overflows could read

```
while (pk < lim)
{
  LOCAL_HIREMAINDER;
  ...
  pk = mulll(pk, p); if (hiremainder) break;
}
```

For most architectures, the functions mentioned below are really chunks of inlined assembler code, and the above 'global' variables are actually local register values.

`ulong addll(ulong x, ulong y)` adds `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong addllx(ulong x, ulong y)` adds `overflow` to the sum of the `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong subll(ulong x, ulong y)` subtracts `x` and `y`, returns the lower `BITS_IN_LONG` bits and put the carry (borrow) bit into `overflow`.

`ulong subllx(ulong x, ulong y)` subtracts `overflow` from the difference of `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry (borrow) bit into `overflow`.

`int bfffo(ulong x)` returns the number of leading zero bits in `x`. That is, the number of bit positions by which it would have to be shifted left until its leftmost bit first becomes equal to 1, which can be between 0 and `BITS_IN_LONG` -1 for nonzero `x`. When `x` is 0, the result is undefined.

`ulong mulll(ulong x, ulong y)` multiplies `x` by `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong addmul(ulong x, ulong y)` adds `hiremainder` to the product of `x` and `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong divll(ulong x, ulong y)` returns the quotient of $(\texttt{hiremainder} * 2^{\texttt{BITS\_IN\_LONG}}) + \texttt{x}$ by `y` and stores the remainder into `hiremainder`. An error occurs if the quotient cannot be represented by an `ulong`, i.e. if initially `hiremainder` $\geq$ `y`.

Obsolete routines. Those functions are awkward and no longer used; they are only provided for backward compatibility:

`ulong shiftl(ulong x, ulong y)` returns x shifted left by y bits, i.e. x `<<` y, where we assume that $0 \leq y \leq$ `BITS_IN_LONG`. The global variable `hiremainder` receives the bits that were shifted out, i.e. x `>>` (`BITS_IN_LONG` $- y$).

`ulong shiftlr(ulong x, ulong y)` returns x shifted right by y bits, i.e. x `>>` y, where we assume that $0 \leq y \leq$ `BITS_IN_LONG`. The global variable `hiremainder` receives the bits that were shifted out, i.e. x `<<` (`BITS_IN_LONG` $- y$).

6.1.2 Modular kernel. The following routines are not part of the level 0 kernel per se, but implement modular operations on words in terms of the above. They are written so that no overflow may occur. Let $m \geq 1$ be the modulus; all operands representing classes modulo m are assumed to belong to $[0, m-1]$. The result may be wrong for a number of reasons otherwise: it may not be reduced, overflow can occur, etc.

`int odd(ulong x)` returns 1 if x is odd, and 0 otherwise.

`int both_odd(ulong x, ulong y)` returns 1 if x and y are both odd, and 0 otherwise.

`ulong invmod2BIL(ulong x)` returns the smallest positive representative of x^{-1} mod $2^{\texttt{BITS\_IN\_LONG}}$, assuming x is odd.

`ulong Fl_add(ulong x, ulong y, ulong m)` returns the smallest positive representative of $x+y$ modulo m.

`ulong Fl_neg(ulong x, ulong m)` returns the smallest positive representative of $-x$ modulo m.

`ulong Fl_sub(ulong x, ulong y, ulong m)` returns the smallest positive representative of $x-y$ modulo m.

`long Fl_center(ulong x, ulong m, ulong mo2)` returns the representative in $]-m/2, m/2]$ of x modulo m. Assume $0 \leq x < m$ and `mo2` $= m >> 1$.

`ulong Fl_mul(ulong x, ulong y, ulong m)` returns the smallest positive representative of xy modulo m.

`ulong Fl_double(ulong x, ulong m)` returns $2x$ modulo m.

`ulong Fl_triple(ulong x, ulong m)` returns $3x$ modulo m.

`ulong Fl_sqr(ulong x, ulong m)` returns the smallest positive representative of x^2 modulo m.

`ulong Fl_inv(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m. If x is not invertible mod m, raise an exception.

`ulong Fl_invsafe(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m. If x is not invertible mod m, return 0 (which is ambiguous if $m = 1$).

`ulong Fl_div(ulong x, ulong y, ulong m)` returns the smallest positive representative of xy^{-1} modulo m. If y is not invertible mod m, raise an exception.

`ulong Fl_powu(ulong x, ulong n, ulong m)` returns the smallest positive representative of x^n modulo m.

`ulong Fl_sqrt(ulong x, ulong p)` returns the square root of `x` modulo `p` (smallest positive representative). Assumes `p` to be prime, and `x` to be a square modulo `p`.

`ulong Fl_order(ulong a, ulong o, ulong p)` returns the order of the `Fp a`. It is assumed that `o` is a multiple of the order of `a`, 0 being allowed (no non-trivial information).

`ulong random_Fl(ulong p)` returns a pseudo-random integer uniformly distributed in $0, 1, \ldots p-1$.

`ulong pgener_Fl(ulong p)` returns the smallest primitive root modulo `p`, assuming `p` is prime.

`ulong pgener_Zl(ulong p)` returns the smallest primitive root modulo p^k, $k > 1$, assuming p is an odd prime.

`ulong pgener_Fl_local(ulong p, GEN L)`, see `gener_Fp_local`, `L` is an `Flv`.

6.1.3 Switching between Fl_xxx and standard operators.

Even though the `Fl_xxx` routines are efficient, they are slower than ordinary `long` operations, using the standard `+`, `%`, etc. operators. The following macro is used to choose in a portable way the most efficient functions for given operands:

`int SMALL_ULONG(ulong p)` true if $2p^2 < 2^{\texttt{BITS\_IN\_LONG}}$. In that case, it is possible to use ordinary operators efficiently. If $p < 2^{\texttt{BITS\_IN\_LONG}}$, one may still use the `Fl_xxx` routines. Otherwise, one must use generic routines. For instance, the scalar product of the `GEN`s x and y mod p could be computed as follows.

```
long i, l = lg(x);
if (lgefint(p) > 3)
{ /* arbitrary */
  GEN s = gen_0;
  for (i = 1; i < l; i++) s = addii(s, mulii(gel(x,i), gel(y,i)));
  return modii(s, p).
}
else
{
  ulong s = 0, pp = itou(p);
  x = ZV_to_Flv(x, pp);
  y = ZV_to_Flv(y, pp);
  if (SMALL_ULONG(pp))
  { /* very small */
    for (i = 1; i < l; i++)
    {
      s += x[i] * y[i];
      if (s & HIGHBIT) s %= pp;
    }
    s %= pp;
  }
  else
  { /* small */
    for (i = 1; i < l; i++)
      s = Fl_add(s, Fl_mul(x[i], y[i], pp), pp);
  }
  return utoi(s);
}
```

In effect, we have three versions of the same code: very small, small, and arbitrary inputs. The very small and arbitrary variants use lazy reduction and reduce only when it becomes necessary: when overflow might occur (very small), and at the very end (very small, arbitrary).

6.2 Level 1 kernel (operations on longs, integers and reals).

Note. Some functions consist of an elementary operation, immediately followed by an assignment statement. They will be introduced as in the following example:

`GEN gadd[z](GEN x, GEN y[, GEN z])` followed by the explicit description of the function

`GEN gadd(GEN x, GEN y)`

which creates its result on the stack, returning a `GEN` pointer to it, and the parts in brackets indicate that there exists also a function

`void gaddz(GEN x, GEN y, GEN z)`

which assigns its result to the pre-existing object `z`, leaving the stack unchanged. These assignment variants are kept for backward compatibility but are inefficient: don't use them.

6.2.1 Creation.

`GEN cgeti(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetipos(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to 1.

`GEN cgetineg(long n)` allocates memory on the PARI stack for a negative `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to -1.

`GEN cgetr(long n)` allocates memory on the PARI stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_REAL)`.

`GEN cgetc(long n)` allocates memory on the PARI stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s of length `n`.

`GEN real_1(long prec)` create a `t_REAL` equal to 1 to `prec` words of accuracy.

`GEN real_m1(long prec)` create a `t_REAL` equal to -1 to `prec` words of accuracy.

`GEN real_0_bit(long bit)` create a `t_REAL` equal to 0 with exponent $-$`bit`.

`GEN real_0(long prec)` is a shorthand for

```
real_0_bit( -bit_accuracy(prec) )
```

`GEN int2n(long n)` creates a `t_INT` equal to `1<<n` (i.e 2^n if $n \geq 0$, and 0 otherwise).

`GEN int2u(ulong n)` creates a `t_INT` equal to 2^n.

`GEN real2n(long n, long prec)` create a `t_REAL` equal to 2^n to `prec` words of accuracy.

`GEN real_m2n(long n, long prec)` create a `t_REAL` equal to -2^n to `prec` words of accuracy.

`GEN strtoi(char *s)` convert the character string `s` to a non-negative `t_INT`. The string `s` consists exclusively of digits: no leading sign, no whitespace. Leading zeroes are discarded.

`GEN strtor(char *s, long prec)` convert the character string `s` to a non-negative `t_REAL` of precision `prec`. The string `s` consists exclusively of digits and optional decimal point and exponent (`e` or `E`): no leading sign, no whitespace. Leading zeroes are discarded.

6.2.2 Assignment. In this section, the `z` argument in the `z`-functions must be of type `t_INT` or `t_REAL`.

`void mpaff(GEN x, GEN z)` assigns `x` into `z` (where `x` and `z` are `t_INT` or `t_REAL`). Assumes that `lg(z)` > 2.

`void affii(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_INT` `z`.

`void affir(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affiz(GEN x, GEN z)` assigns `t_INT` `x` into `t_INT` or `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affsi(long s, GEN z)` assigns the `long` `s` into the `t_INT` `z`. Assumes that `lg(z)` > 2.

`void affsr(long s, GEN z)` assigns the `long` `s` into the `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affsz(long s, GEN z)` assigns the `long` `s` into the `t_INT` or `t_REAL` `z`. Assumes that `lg(z)` $>$ 2.

`void affui(ulong u, GEN z)` assigns the `ulong` `u` into the `t_INT` `z`. Assumes that `lg(z)` > 2.

`void affur(ulong u, GEN z)` assigns the `ulong` `u` into the `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affrr(GEN x, GEN z)` assigns the `t_REAL` `x` into the `t_REAL` `z`.

`void affgr(GEN x, GEN z)` assigns the scalar `x` into the `t_REAL` `z`, if possible.

The function `affrs` and `affri` do not exist. So don't use them.

`void affrr_fixlg(GEN y, GEN z)` a variant of `affrr`. First shorten z so that it is no longer than y, then assigns y to z. This is used in the following scenario: room is reserved for the result but, due to cancellation, fewer words of accuracy are available than had been anticipated; instead of appending meaningless 0s to the mantissa, we store what was actually computed.

Note that shortening z is not quite straightforward, since `setlg(z, ly)` would leave garbage on the stack, which `gerepile` might later inspect. It is done using

`void fixlg(GEN z, long ly)` see `stackdummy` and the examples that follow.

6.2.3 Copy.

`GEN icopy(GEN x)` copy relevant words of the `t_INT` `x` on the stack: the length and effective length of the copy are equal.

`GEN rcopy(GEN x)` copy the `t_REAL` `x` on the stack.

`GEN leafcopy(GEN x)` copy the leaf `x` on the stack (works in particular for `t_INT`s and `t_REAL`s). Contrary to `icopy`, `leafcopy` preserves the original length of a `t_INT`. The obsolete form `GEN mpcopy(GEN x)` is still provided for backward compatibility.

This function also works on recursive types, copying them as if they were leaves, i.e. making a shallow copy in that case: the components of the copy point to the same data as the component of the source; see also `shallowcopy`.

`GEN leafcopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a leaf and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

`GEN icopyspec(GEN x, long nx)` copy the `nx` words `x[2]`, ..., `x[nx+1]` to make up a new `t_INT`. Set the sign to 1.

6.2.4 Conversions.

`GEN itor(GEN x, long prec)` converts the `t_INT x` to a `t_REAL` of length `prec` and return the latter. Assumes that `prec` > 2.

`long itos(GEN x)` converts the `t_INT x` to a `long` if possible, otherwise raise an exception.

`long itos_or_0(GEN x)` converts the `t_INT x` to a `long` if possible, otherwise return 0.

`int is_bigint(GEN n)` true if `itos(n)` would succeed.

`int is_bigint_lg(GEN n, long l)` true if `itos(n)` would succeed. Assumes `lgefint(n)` is equal to `l`.

`ulong itou(GEN x)` converts the `t_INT |x|` to an `ulong` if possible, otherwise raise an exception.

`long itou_or_0(GEN x)` converts the `t_INT |x|` to an `ulong` if possible, otherwise return 0.

`GEN stoi(long s)` creates the `t_INT` corresponding to the `long s`.

`GEN stor(long s, long prec)` converts the `long s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec` > 2.

`GEN utoi(ulong s)` converts the `ulong s` into a `t_INT` and return the latter.

`GEN utoipos(ulong s)` converts the *non-zero* `ulong s` into a `t_INT` and return the latter.

`GEN utoineg(ulong s)` converts the *non-zero* `ulong s` into the `t_INT` $-s$ and return the latter.

`GEN utor(ulong s, long prec)` converts the `ulong s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec` > 2.

`GEN rtor(GEN x, long prec)` converts the `t_REAL x` to a `t_REAL` of length `prec` and return the latter. If `prec` $<$ `lg(x)`, round properly. If `prec` $>$ `lg(x)`, pad with zeroes. Assumes that `prec` > 2.

The following function is also available as a special case of `mkintn`:

`GEN uu32toi(ulong a, ulong b)` returns the `GEN` equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

`GEN uutoi(ulong a, ulong b)` returns the `GEN` equal to $2^{\texttt{BITS\_IN\_LONG}}a + b$.

`GEN uutoineg(ulong a, ulong b)` returns the `GEN` equal to $-(2^{\texttt{BITS\_IN\_LONG}}a + b)$.

6.2.5 Integer parts. The following four functions implement the conversion from `t_REAL` to `t_INT` using standard rounding modes. Contrary to usual semantics (complement the mantissa with an infinite number of 0), they will raise an error *precision loss in truncation* if the `t_REAL` represents a range containing more than one integer.

`GEN ceilr(GEN x)` smallest integer larger or equal to the `t_REAL x` (i.e. the `ceil` function).

`GEN floorr(GEN x)` largest integer smaller or equal to the `t_REAL x` (i.e. the `floor` function).

`GEN roundr(GEN x)` rounds the `t_REAL x` to the nearest integer (towards $+\infty$ in case of tie).

`GEN truncr(GEN x)` truncates the `t_REAL x` (not the same as `floorr` if `x` is negative).

The following four function are analogous, but can also treat the trivial case when the argument is a `t_INT`:

`GEN mpceil(GEN x)` as `ceilr` except that `x` may be a `t_INT`.

`GEN mpfloor(GEN x)` as `floorr` except that `x` may be a `t_INT`.

`GEN mpround(GEN x)` as `roundr` except that `x` may be a `t_INT`.

`GEN mptrunc(GEN x)` as `truncr` except that `x` may be a `t_INT`.

`GEN diviiround(GEN x, GEN y)` if `x` and `y` are `t_INT`s, returns the quotient `x/y` of `x` and `y`, rounded to the nearest integer. If `x/y` falls exactly halfway between two consecutive integers, then it is rounded towards $+\infty$ (as for `roundr`).

`GEN ceil_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the smallest integer which is larger than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.) Note that `gceil` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN floor_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the largest integer which is smaller than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.) Note that `gfloor` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN trunc_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the integer with the largest absolute value, which is closer to 0 than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.)

`GEN roundr_safe(GEN x)` rounds the `t_REAL` `x` to the nearest integer (towards $+\infty$). Complement the mantissa with an infinite number of 0 before rounding, hence never raise an exception.

6.2.6 2-adic valuations and shifts.

`long vals(long s)` 2-adic valuation of the `long` `s`. Returns -1 if `s` is equal to 0.

`long vali(GEN x)` 2-adic valuation of the `t_INT` `x`. Returns -1 if `x` is equal to 0.

`GEN mpshift(GEN x, long n)` shifts the `t_INT` or `t_REAL` `x` by `n`. If `n` is positive, this is a left shift, i.e. multiplication by $2^{\mathtt{n}}$. If `n` is negative, it is a right shift by $-\mathtt{n}$, which amounts to the truncation of the quotient of `x` by $2^{-\mathtt{n}}$.

`GEN shifti(GEN x, long n)` shifts the `t_INT` x by n.

`GEN shiftr(GEN x, long n)` shifts the `t_REAL` x by n.

`void shiftr_inplace(GEN x, long n)` shifts the `t_REAL` x by n, in place.

`GEN trunc2nr(GEN x, long n)` given a `t_REAL` x, returns `truncr(shiftr(x,n))`, but faster, without leaving garbage on the stack and never raising a *precision loss in truncation* error. Called by `gtrunc2n`.

`GEN trunc2nr_lg(GEN x, long lx, long n)` given a `t_REAL` x, returns `trunc2nr(x,n)`, pretending that the length of x is `lx`, which must be $\leq \mathtt{lg}(x)$.

`GEN mantissa2nr(GEN x, long n)` given a `t_REAL` x, returns the mantissa of $x2^n$ (disregards the exponent of x). Equivalent to

```
trunc2nr(x, n-expo(x)+bit_prec(x)-1)
```

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z, and sets `*e` to the exponent $\mathtt{bit\_accuracy}(\mathtt{lg}(\mathtt{z})) - 1 - \mathtt{expo}(z)$, so that $z = m/2^e$.

Low-level. In the following two functions, s(ource) and t(arget) need not be valid `GEN`s (in practice, they usually point to some part of a `t_REAL` mantissa): they are considered as arrays of words representing some mantissa, and we shift globally s by $n > 0$ bits, storing the result in t. We assume that $m \leq M$ and only access $s[m], s[m+1], \dots s[M]$ (read) and likewise for t (write); we may have $s = t$ but more general overlaps are not allowed. The word f is concatenated to s to supply extra bits.

`void shift_left(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$$s[m], s[m+1], \dots s[M], f$$

left by n bits.

`void shift_right(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$$f, s[m], s[m+1], \dots s[M]$$

right by n bits.

6.2.7 Integer valuation. For integers x and p, such that $x \neq 0$ and $|p| > 1$, we define $v_p(x)$ to be the largest integer exponent e such that p^e divides x. If p is prime, this is the ordinary valuation of x at p.

`long Z_pvalrem(GEN x, GEN p, GEN *r)` applied to `t_INT`s `x` $\neq 0$ and `p`, $|$`p`$| > 1$, returns $e :=$ $v_p(x)$ The quotient `x`$/$`p`^{e} is returned in `*r`. If |`p`| is a prime, `*r` is the prime-to-`p` part of `x`.

`long Z_pval(GEN x, GEN p)` as `Z_pvalrem` but only returns $v_p(x)$.

`long Z_lvalrem(GEN x, ulong p, GEN *r)` as `Z_pvalrem`, except that `p` is an `ulong` (`p` > 1).

`long Z_lvalrem_stop(GEN *x, ulong p, int *stop)` returns $e := v_p(x)$ and replaces x by x/p^e. Set `stop` to 1 if the new value of x is $< p^2$ (and 0 otherwise). To be used when trial dividing x by successive primes: the `stop` condition is cheaply tested while testing whether p divides x (is the quotient less than p?), and allows to decide that n is prime if no prime $< p$ divides n. Not memory-clean.

`long Z_lval(GEN x, ulong p)` as `Z_pval`, except that `p` is an `ulong` (`p` > 1).

`long u_lvalrem(ulong x, ulong p, ulong *r)` as `Z_pvalrem`, except the inputs/outputs are now `ulong`s.

`long u_lvalrem_stop(ulong *n, ulong p, int *stop)` as `Z_pvalrem_stop`.

`long u_pvalrem(ulong x, GEN p, ulong *r)` as `Z_pvalrem`, except `x` and `r` are now `ulong`s.

`long u_lval(ulong x, ulong p)` as `Z_pval`, except the inputs are now `ulong`s.

`long u_pval(ulong x, GEN p)` as `Z_pval`, except `x` is now an `ulong`.

`long z_lval(long x, ulong p)` as `u_lval`, for signed `x`.

`long z_lvalrem(long x, ulong p)` as `u_lvalrem`, for signed `x`.

`long z_pval(long x, GEN p)` as `Z_pval`, except `x` is now a `long`.

`long z_pvalrem(long x, GEN p)` as `Z_pvalrem`, except `x` is now a `long`.

`long Q_pval(GEN x, GEN p)` valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.

`long factorial_lval(ulong n, ulong p)` returns $v_p(n!)$, assuming p is prime.

The following convenience functions generalize `Z_pval` and its variants to "containers" (`ZV` and `ZX`):

`long ZV_pvalrem(GEN x, GEN p, GEN *r)` x being a `ZV` (a vector of `t_INT`s), return the min v of the valuations of its components and set `*r` to x/p^v. Infinite loop if x is the zero vector. This function is not stack clean.

`long ZV_pval(GEN x, GEN p)` as `ZV_pvalrem` but only returns the "valuation".

`int ZV_Z_dvd(GEN x, GEN p)` returns 1 if p divides all components of x and 0 otherwise. Faster than testing `ZV_pval(x,p) >= 1`.

`long ZV_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_pvalrem`, except that `p` is an `ulong` (`p` > 1). This function is not stack-clean.

`long ZV_lval(GEN x, ulong p)` as `ZV_pval`, except that `p` is an `ulong` (`p` > 1).

`long ZX_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZX` x (a `t_POL` with `t_INT` coefficients). This function is not stack-clean.

`long ZX_pval(GEN x, GEN p)` as `ZV_pval` for a `ZX` x.

`long ZX_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_lvalrem`, a `ZX` x. This function is not stack-clean.

`long ZX_lval(GEN x, ulong p)` as `ZX_pval`, except that `p` is an `ulong` (`p` > 1).

6.2.8 Generic unary operators. Let "*op*" be a unary operation among

- **neg**: negation ($-x$).
- **abs**: absolute value ($|x|$).
- **sqr**: square (x^2).

The names and prototypes of the low-level functions corresponding to *op* are as follows. The result is of the same type as `x`.

`GEN` *op*`i(GEN x)` creates the result of *op* applied to the `t_INT` `x`.

`GEN` *op*`r(GEN x)` creates the result of *op* applied to the `t_REAL` `x`.

`GEN mp`*op*`(GEN x)` creates the result of *op* applied to the `t_INT` or `t_REAL` `x`.

Complete list of available functions:

`GEN absi(GEN x)`, `GEN absr(GEN x)`, `GEN mpabs(GEN x)`

`GEN negi(GEN x)`, `GEN negr(GEN x)`, `GEN mpneg(GEN x)`

`GEN sqri(GEN x)`, `GEN sqrr(GEN x)`, `GEN mpsqr(GEN x)`

`GEN absi_shallow(GEN x)` x being a `t_INT`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `negi(`x`)` otherwise.

`GEN mpabs_shallow(GEN x)` x being a `t_INT` or a `t_REAL`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `mpneg(`x`)` otherwise.

Some miscellaneous routines:

`GEN sqrs(long x)` returns x^2.

`GEN sqru(ulong x)` returns x^2.

6.2.9 Comparison operators.

`long minss(long x, long y)`

`ulong minuu(ulong x, ulong y)`

`double mindd(double x, double y)` returns the `min` of x and y.

`long maxss(long x, long y)`

`ulong maxuu(ulong x, ulong y)`

`double maxdd(double x, double y)` returns the `max` of x and y.

`int mpcmp(GEN x, GEN y)` compares the `t_INT` or `t_REAL` `x` to the `t_INT` or `t_REAL` `y`. The result is the sign of `x` $-$ `y`.

`int cmpii(GEN x, GEN y)` compares the `t_INT` `x` to the `t_INT` `y`.

`int cmpir(GEN x, GEN y)` compares the `t_INT` `x` to the `t_REAL` `y`.

`int cmpis(GEN x, long s)` compares the `t_INT` `x` to the `long` `s`.

`int cmpsi(long s, GEN x)` compares the `long` `s` to the `t_INT` `x`.

`int cmpsr(long s, GEN x)` compares the `long` `s` to the `t_REAL` `x`.

`int cmpri(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_INT` `y`.

`int cmprr(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_REAL` `y`.

`int cmprs(GEN x, long s)` compares the `t_REAL` `x` to the `long` `s`.

`int equalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if `x` $=$ `y`, 0 otherwise.

`int equalrr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is 1 if `x` $=$ `y`, 0 otherwise. Equality is decided according to the following rules: all real zeroes are equal, and different from a non-zero real; two non-zero reals are equal if all their digits coincide up to the length of the shortest of the two, and the remaining words in the mantissa of the longest are all 0.

`int equalsi(long s, GEN x)`

`int equalis(GEN x, long s)` compare the `t_INT` `x` and the `long` `s`. The result is 1 if `x` $=$ `y`, 0 otherwise.

The remaining comparison operators disregard the sign of their operands:

`int equalui(ulong s, GEN x)`

`int equaliu(GEN x, ulong s)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`. The result is 1 if |`x`| $=$ `y`, 0 otherwise.

`int cmpui(ulong u, GEN x)`

`int cmpiu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`.

`int absi_cmp(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is the sign of |`x`| $-$ |`y`|.

`int absi_equal(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if |`x`| $=$ |`y`|, 0 otherwise.

`int absr_cmp(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is the sign of |`x`| $-$ |`y`|.

`int absrnz_equal2n(GEN x)` tests whether a non-zero `t_REAL x` is equal to $\pm 2^e$ for some integer e.

`int absrnz_equal1(GEN x)` tests whether a non-zero `t_REAL x` is equal to ± 1.

6.2.10 Generic binary operators. The operators in this section have arguments of C-type `GEN`, `long`, and `ulong`, and only `t_INT` and `t_REAL GEN`s are allowed. We say an argument is a real type if it is a `t_REAL GEN`, and an integer type otherwise. The result is always a `t_REAL` unless both `x` and `y` are integer types.

Let "*op*" be a binary operation among

• **add**: addition (`x + y`).

• **sub**: subtraction (`x - y`).

• **mul**: multiplication (`x * y`).

• **div**: division (`x / y`). In the case where `x` and `y` are both integer types, the result is the Euclidean quotient, where the remainder has the same sign as the dividend `x`. It is the ordinary division otherwise. A division-by-0 error occurs if `y` is equal to 0.

The last two generic operations are defined only when arguments have integer types; and the result is a `t_INT`:

• **rem**: remainder ("`x % y`"). The result is the Euclidean remainder corresponding to `div`, i.e. its sign is that of the dividend `x`.

• **mod**: true remainder (`x % y`). The result is the true Euclidean remainder, i.e. non-negative and less than the absolute value of `y`.

Important technical note. The rules given above fixing the output type (to `t_REAL` unless both inputs are integer types) are subtly incompatible with the general rules obeyed by PARI's generic functions, such as `gmul` or `gdiv` for instance: the latter return a result containing as much information as could be deduced from the inputs, so it is not true that if x is a `t_INT` and y a `t_REAL`, then `gmul(x,y)` is always the same as `mulir(x,y)`. The exception is $x = 0$, in that case we can deduce that the result is an exact 0, so `gmul` returns `gen_0`, while `mulir` returns a `t_REAL` 0. Specifically, the one resulting from the conversion of `gen_0` to a `t_REAL` of precision `precision(y)`, multiplied by y; this determines the exponent of the real 0 we obtain.

The reason for the discrepancy between the two rules is that we use the two sets of functions in different contexts: generic functions allow to write high-level code forgetting about types, letting PARI return results which are sensible and as simple as possible; type specific functions are used in kernel programming, where we do care about types and need to maintain strict consistency: it is much easier to compute the types of results when they are determined from the types of the inputs only (without taking into account further arithmetic properties, like being non-0).

The names and prototypes of the low-level functions corresponding to *op* are as follows. In this section, the `z` argument in the `z`-functions must be of type `t_INT` when no `r` or `mp` appears in the argument code (no `t_REAL` operand is involved, only integer types), and of type `t_REAL` otherwise.

`GEN mp`*op*`[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT` or `t_REAL x` and `y`. The function `mpdivz` does not exist (its semantic would change drastically depending on the type of the `z` argument), and neither do `mprem[z]` nor `mpmod[z]` (specific to integers).

`GEN` *op*`si[z](long s, GEN x[, GEN z])` applies *op* to the `long s` and the `t_INT x`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`sr[z](long s, GEN x[, GEN z])` applies *op* to the `long s` and the `t_REAL x`.

`GEN` *op*`ss[z](long s, long t[, GEN z])` applies *op* to the longs `s` and `t`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`ii[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT`s `x` and `y`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`ir[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT x` and the `t_REAL y`.

`GEN` *op*`is[z](GEN x, long s[, GEN z])` applies *op* to the `t_INT x` and the `long s`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`ri[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_REAL x` and the `t_INT y`.

`GEN` *op*`rr[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_REAL`s `x` and `y`.

`GEN` *op*`rs[z](GEN x, long s[, GEN z])` applies *op* to the `t_REAL x` and the `long s`.

Some miscellaneous routines:

`long expu(ulong x)` assuming $x > 0$, returns the binary exponent of the real number equal to x. This is a special case of `gexpo`.

`GEN adduu(ulong x, ulong y)`

`GEN addiu(GEN x, ulong y)`

`GEN addui(ulong x, GEN y)` adds `x` and `y`.

`GEN subuu(ulong x, ulong y)`

`GEN subiu(GEN x, ulong y)`

`GEN subui(ulong x, GEN y)` subtracts `x` by `y`.

`GEN muluu(ulong x, ulong y)` multiplies `x` by `y`.

`GEN mului(ulong x, GEN y)` multiplies `x` by `y`.

`GEN muluui(ulong x, ulong y, GEN z)` return xyz.

`GEN muliu(GEN x, ulong y)` multiplies `x` by `y`.

`void addumului(ulong a, ulong b, GEN x)` return $a + b|X|$.

`GEN addmuliu(GEN x, GEN y, ulong u)` returns $x + yu$.

`GEN addmulii(GEN x, GEN y, GEN z)` returns $x + yz$.

`GEN addmulii_inplace(GEN x, GEN y, GEN z)` returns $x+yz$, but returns x itself and not a copy if $yz = 0$. Not suitable for `gerepile` or `gerepileupto`.

`GEN addmuliu_inplace(GEN x, GEN y, ulong u)` returns $x + yu$, but returns x itself and not a copy if $yu = 0$. Not suitable for `gerepile` or `gerepileupto`.

`GEN submuliu_inplace(GEN x, GEN y, ulong u)` returns $x - yu$, but returns x itself and not a copy if $yu = 0$. Not suitable for `gerepile` or `gerepileupto`.

`GEN lincombii(GEN u, GEN v, GEN x, GEN y)` returns $ux + vy$.

`GEN mulsubii(GEN y, GEN z, GEN x)` returns $yz - x$.

`GEN submulii(GEN x, GEN y, GEN z)` returns $x - yz$.

`GEN submuliu(GEN x, GEN y, ulong u)` returns $x - yu$.

`GEN mulu_interval(ulong a, ulong b)` returns $a(a+1)\cdots b$, assuming that $a \leq b$. Very inefficient when $a = 0$.

`GEN invr(GEN x)` returns the inverse of the non-zero `t_REAL` x.

`GEN truedivii(GEN x, GEN y)` returns the true Euclidean quotient (with non-negative remainder less than $|y|$).

`GEN truedivis(GEN x, long y)` returns the true Euclidean quotient (with non-negative remainder less than $|y|$).

`GEN truedivsi(long x, GEN y)` returns the true Euclidean quotient (with non-negative remainder less than $|y|$).

`GEN centermodii(GEN x, GEN y, GEN y2)`, given `t_INT`s `x`, `y`, returns z congruent to `x` modulo `y`, such that $-\mathtt{y}/2 \leq z < \mathtt{y}/2$. The function requires an extra argument `y2`, such that `y2 = shifti(y, -1)`. (In most cases, `y` is constant for many reductions and `y2` need only be computed once.)

`GEN remi2n(GEN x, long n)` returns `x` mod 2^n.

`GEN addii_sign(GEN x, long sx, GEN y, long sy)` add the `t_INT`s x and y as if their signs were `sx` and `sy`.

`GEN addir_sign(GEN x, long sx, GEN y, long sy)` add the `t_INT` x and the `t_REAL` y as if their signs were `sx` and `sy`.

`GEN addrr_sign(GEN x, long sx, GEN y, long sy)` add the `t_REAL`s x and y as if their signs were `sx` and `sy`.

`GEN addsi_sign(long x, GEN y, long sy)` add x and the `t_INT` y as if its sign was `sy`.

`GEN addui_sign(ulong x, GEN y, long sy)` add x and the `t_INT` y as if its sign was `sy`.

6.2.11 Exact division and divisibility.

`GEN diviiexact(GEN x, GEN y)` returns the Euclidean quotient `x`/`y`, assuming `y` divides `x`. Uses Jebelean algorithm (Jebelean-Krandick bidirectional exact division is not implemented).

`GEN diviuexact(GEN x, ulong y)` returns the Euclidean quotient `x`/`y`, assuming `y` divides `x` and `y` is non-zero.

`GEN diviuuexact(GEN x, ulong y, ulong z)` returns the Euclidean quotient $x/(yz)$, assuming yz divides x and $yz \neq 0$.

The following routines return 1 (true) if `y` divides `x`, and 0 otherwise. (Error if y is 0, even if x is 0.) All `GEN` are assumed to be `t_INT`s:

`int dvdii(GEN x, GEN y)`, `int dvdis(GEN x, long y)`, `int dvdiu(GEN x, ulong y)`,

`int dvdsi(long x, GEN y)`, `int dvdui(ulong x, GEN y)`.

The following routines return 1 (true) if `y` divides `x`, and in that case assign the quotient to `z`; otherwise they return 0. All `GEN` are assumed to be `t_INT`s:

`int dvdiiz(GEN x, GEN y, GEN z)`, `int dvdisz(GEN x, long y, GEN z)`.

`int dvdiuz(GEN x, ulong y, GEN z)` if `y` divides `x`, assigns the quotient |`x`|/`y` to `z` and returns 1 (true), otherwise returns 0 (false).

6.2.12 Division with integral operands and `t_REAL` result.

`GEN rdivii(GEN x, GEN y, long prec)`, assuming x and y are both of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdiviiz(GEN x, GEN y, GEN z)`, assuming x and y are both of type `t_INT`, and z is a `t_REAL`, assign the quotient x/y to z.

`GEN rdivis(GEN x, long y, long prec)`, assuming `x` is of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdivsi(long x, GEN y, long prec)`, assuming `y` is of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdivss(long x, long y, long prec)`, return the quotient x/y as a `t_REAL` of precision `prec`.

6.2.13 Division with remainder. The following functions return two objects, unless specifically asked for only one of them — a quotient and a remainder. The quotient is returned and the remainder is returned through the variable whose address is passed as the `r` argument. The term *true Euclidean remainder* refers to the non-negative one (`mod`), and *Euclidean remainder* by itself to the one with the same sign as the dividend (`rem`). All `GEN`s, whether returned directly or through a pointer, are created on the stack.

`GEN dvmdii(GEN x, GEN y, GEN *r)` returns the Euclidean quotient of the `t_INT` `x` by a `t_INT` `y` and puts the remainder into `*r`. If `r` is equal to `NULL`, the remainder is not created, and if `r` is equal to `ONLY_REM`, only the remainder is created and returned. In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`. The remainder is always of the sign of the dividend `x`. If the remainder is 0 set `r = gen_0`.

`void dvmdiiz(GEN x, GEN y, GEN z, GEN t)` assigns the Euclidean quotient of the `t_INT`s `x` and `y` into the `t_INT` `z`, and the Euclidean remainder into the `t_INT` `t`.

Analogous routines `dvmdis[z]`, `dvmdsi[z]`, `dvmdss[z]` are available, where `s` denotes a `long` argument. But the following routines are in general more flexible:

`long sdivss_rem(long s, long t, long *r)` computes the Euclidean quotient and remainder of the longs `s` and `t`. Puts the remainder into `*r`, and returns the quotient. The remainder is of the sign of the dividend `s`, and has strictly smaller absolute value than `t`.

`long sdivsi_rem(long s, GEN x, long *r)` computes the Euclidean quotient and remainder of the `long` `s` by the `t_INT` `x`. As `sdivss_rem` otherwise.

`long sdivsi(long s, GEN x)` as `sdivsi_rem`, without remainder.

`GEN divis_rem(GEN x, long s, long *r)` computes the Euclidean quotient and remainder of the `t_INT` `x` by the `long` `s`. As `sdivss_rem` otherwise.

`GEN diviu_rem(GEN x, ulong s, ulong *r)` computes the Euclidean quotient and remainder of *absolute value* of the `t_INT` `x` by the `ulong` `s`. As `sdivss_rem` otherwise.

`ulong udiviu_rem(GEN n, ulong d, ulong *r)` as `diviu_rem`, assuming that $|n|/d$ fits into an `ulong`.

`ulong udivui_rem(ulong x, GEN y, ulong *rem)` computes the Euclidean quotient and remainder of x by y. As `sdivss_rem` otherwise.

`ulong udivuu_rem(ulong x, ulong y, ulong *rem)` computes the Euclidean quotient and remainder of x by y. As `sdivss_rem` otherwise.

`GEN divsi_rem(long s, GEN y, long *r)` computes the Euclidean quotient and remainder of the `t_long s` by the `GEN y`. As `sdivss_rem` otherwise.

`GEN divss_rem(long x, long y, long *r)` computes the Euclidean quotient and remainder of the `t_long x` by the `long y`. As `sdivss_rem` otherwise.

`GEN truedvmdii(GEN x, GEN y, GEN *r)`, as `dvmdii` but with a non-negative remainder.

`GEN truedvmdis(GEN x, long y, GEN *z)`, as `dvmdis` but with a non-negative remainder.

`GEN truedvmdsi(long x, GEN y, GEN *z)`, as `dvmdsi` but with a non-negative remainder.

6.2.14 Modulo to longs. The following variants of `modii` do not clutter the stack:

`long smodis(GEN x, long y)` computes the true Euclidean remainder of the `t_INT x` by the `long y`. This is the non-negative remainder, not the one whose sign is the sign of `x` as in the `div` functions.

`long smodss(long x, long y)` computes the true Euclidean remainder of the `long x` by a `long y`.

`ulong umodiu(GEN x, ulong y)` computes the true Euclidean remainder of the `t_INT x` by the `ulong y`.

`ulong umodui(ulong x, GEN y)` computes the true Euclidean remainder of the `ulong x` by the `t_INT |y|`.

The routine `smodsi` does not exist, since it would not always be defined: for a *negative* `x`, if the quotient is ± 1, the result `x + |y|` would in general not fit into a `long`. Use either `umodui` or `modsi`.

6.2.15 Powering, Square root.

`GEN powii(GEN x, GEN n)`, assumes x and n are `t_INT`s and returns x^n.

`GEN powuu(ulong x, ulong n)`, returns x^n.

`GEN powiu(GEN x, ulong n)`, assumes x is a `t_INT` and returns x^n.

`GEN powis(GEN x, long n)`, assumes x is a `t_INT` and returns x^n (possibly a `t_FRAC` if $n < 0$).

`GEN powrs(GEN x, long n)`, assumes x is a `t_REAL` and returns x^n. This is considered as a sequence of `mulrr`, possibly empty: as such the result has type `t_REAL`, even if $n = 0$. Note that the generic function `gpowgs(x,0)` would return `gen_1`, see the technical note in Section 6.2.10.

`GEN powru(GEN x, ulong n)`, assumes x is a `t_REAL` and returns x^n (always a `t_REAL`, even if $n = 0$).

`GEN powruvec(GEN e, ulong n)`. Given a `t_REAL` e, return the vector of all e^i, $1 \leq i \leq n$.

`GEN powrshalf(GEN x, long n)`, assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

`GEN powruhalf(GEN x, ulong n)`, assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

`GEN powrfrac(GEN x, long n, long d)`, assumes x is a `t_REAL` and returns $x^{n/d}$ (always a `t_REAL`, even if $n = 0$).

`GEN powIs(long n)` returns $I^n \in \{1, I, -1, -I\}$ (`t_INT` for even n, `t_COMPLEX` otherwise).

`ulong upowuu(ulong x, ulong n)`, returns x^n when $< 2^{\texttt{BITS\_IN\_LONG}}$, and 0 otherwise (overflow).

`GEN sqrtremi(GEN N, GEN *r)`, returns the integer square root S of the non-negative `t_INT N` (rounded towards 0) and puts the remainder R into `*r`. Precisely, $N = S^2 + R$ with $0 \leq R \leq 2S$. If `r` is equal to `NULL`, the remainder is not created. In the generic case, the remainder is created after the quotient and can be disposed of individually with `cgiv(R)`. If the remainder is 0 set `R = gen_0`.

Uses a divide and conquer algorithm (discrete variant of Newton iteration) due to Paul Zimmermann ("Karatsuba Square Root", INRIA Research Report 3805 (1999)).

`GEN sqrti(GEN N)`, returns the integer square root S of the non-negative `t_INT N` (rounded towards 0). This is identical to `sqrtremi(N, NULL)`.

6.2.16 GCD, extended GCD and LCM.

`long cgcd(long x, long y)` returns the GCD of `x` and `y`.

`ulong ugcd(ulong x, ulong y)` returns the GCD of `x` and `y`.

`long clcm(long x, long y)` returns the LCM of `x` and `y`, provided it fits into a `long`. Silently overflows otherwise.

`GEN gcdii(GEN x, GEN y)`, returns the GCD of the `t_INT`s `x` and `y`.

`GEN lcmii(GEN x, GEN y)`, returns the LCM of the `t_INT`s `x` and `y`.

`GEN bezout(GEN a,GEN b, GEN *u,GEN *v)`, returns the GCD d of `t_INT`s `a` and `b` and sets `u`, `v` to the Bezout coefficients such that $\texttt{au} + \texttt{bv} = d$.

`long cbezout(long a,long b, long *u,long *v)`, returns the GCD d of `a` and `b` and sets `u`, `v` to the Bezout coefficients such that $\texttt{au} + \texttt{bv} = d$.

`GEN ZV_gcdext(GEN A)` given a vector of n integers A, returns $[d, U]$, where d is the GCD of the $A[i]$ and U is a matrix in $\mathrm{GL}_n(\mathbf{Z})$ such that $AU = [0, \ldots, 0, D]$.

6.2.17 Pseudo-random integers. These routine return pseudo-random integers uniformly distributed in some interval. The all use the same underlying generator which can be seeded and restarted using `getrand` and `setrand`.

`void setrand(GEN seed)` reseeds the random number generator using the seed n. The seed is either a technical array output by `getrand` or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by `setrand(1)` twice will generate the exact same output.

`GEN getrand(void)` returns the current value of the seed used by the pseudo-random number generator `random`. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array of type `t_VECSMALL`), and can only be used as an argument to `setrand`.

`ulong pari_rand(void)` returns a random $0 \le x < 2^{\texttt{BITS\_IN\_LONG}}$.

`long random_bits(long k)` returns a random $0 \le x < 2^k$. Assumes that $0 \le k \le$ `BITS_IN_LONG`.

`ulong random_Fl(ulong p)` returns a pseudo-random integer in $0, 1, \ldots p-1$.

`GEN randomi(GEN n)` returns a random `t_INT` between 0 and `n` $- 1$.

`GEN randomr(long prec)` returns a random `t_REAL` in $[0, 1[$, with precision `prec`.

6.2.18 Modular operations. In this subsection, all `GEN`s are `t_INT`

`GEN Fp_red(GEN a, GEN m)` returns `a` modulo `m` (smallest non-negative residue). (This is identical to modii).

`GEN Fp_neg(GEN a, GEN m)` returns $-$`a` modulo `m` (smallest non-negative residue).

`GEN Fp_add(GEN a, GEN b, GEN m)` returns the sum of `a` and `b` modulo `m` (smallest non-negative residue).

`GEN Fp_sub(GEN a, GEN b, GEN m)` returns the difference of `a` and `b` modulo `m` (smallest non-negative residue).

`GEN Fp_center(GEN a, GEN p, GEN pov2)` assuming that `pov2` is `shifti(p,-1)` and that `a` is between 0 and `p` $- 1$ and, returns the representative of `a` in the symmetric residue system.

`GEN Fp_mul(GEN a, GEN b, GEN m)` returns the product of `a` by `b` modulo `m` (smallest non-negative residue).

`GEN Fp_addmul(GEN x, GEN y, GEN z, GEN p)` returns $x + yz$.

`GEN Fp_mulu(GEN a, ulong b, GEN m)` returns the product of `a` by `b` modulo `m` (smallest non-negative residue).

`GEN Fp_muls(GEN a, long b, GEN m)` returns the product of `a` by `b` modulo `m` (smallest non-negative residue).

`GEN Fp_sqr(GEN a, GEN m)` returns `a`^{2} modulo `m` (smallest non-negative residue).

`ulong Fp_powu(GEN x, ulong n, GEN m)` raises `x` to the `n`-th power modulo `m` (smallest non-negative residue). Not memory-clean, but suitable for `gerepileupto`.

`ulong Fp_pows(GEN x, long n, GEN m)` raises `x` to the `n`-th power modulo `m` (smallest non-negative residue). A negative `n` is allowed Not memory-clean, but suitable for `gerepileupto`.

`GEN Fp_pow(GEN x, GEN n, GEN m)` returns `x`$^{\texttt{n}}$ modulo `m` (smallest non-negative residue).

`GEN Fp_inv(GEN a, GEN m)` returns an inverse of `a` modulo `m` (smallest non-negative residue). Raise an error if `a` is not invertible.

`GEN Fp_invsafe(GEN a, GEN m)` as `Fp_inv`, but return `NULL` if `a` is not invertible.

`GEN FpV_inv(GEN x, GEN m)` x being a vector of `t_INT`s, return the vector of inverses of the $x[i]$ mod m. The routine uses Montgomery's trick, and involves a single inversion mod m, plus $3(N-1)$ multiplications for N entries. The routine is not stack-clean: $2N$ integers mod m are left on stack, besides the N in the result.

`GEN Fp_div(GEN a, GEN b, GEN m)` returns the quotient of `a` by `b` modulo `m` (smallest non-negative residue). Raise an error if `b` is not invertible.

`int invmod(GEN a, GEN m, GEN *g)`, return 1 if `a` modulo `m` is invertible, else return 0 and set `g` = gcd(`a`, `m`).

`GEN Fp_log(GEN a, GEN g, GEN ord, GEN p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is currently undefined.

`GEN Fp_order(GEN a, GEN N, GEN p)` returns the order of the `Fp a`. If `N` is non-`NULL`, it is assumed that `N` is a multiple of the order of `a`, as a `t_INT` or a factorization matrix.

`GEN Fp_factored_order(GEN a, GEN N, GEN p)` returns $[o, F]$, where o is the multiplicative order of the `Fp` a in $\mathbf{F}_p^*$, and F is the factorization of o. If `N` is non-`NULL`, it is assumed that `N` is a multiple of the order of `a`, as a `t_INT` or a factorization matrix.

`int Fp_issquare(GEN x, GEN p)` returns 1 if `x` is a square modulo `p`, and 0 otherwise.

`int Fp_ispower(GEN x, GEN n, GEN p)` returns 1 if `x` is an n-th power modulo `p`, and 0 otherwise.

`GEN Fp_sqrt(GEN x, GEN p)` returns a square root of `x` modulo `p` (the smallest non-negative residue), where `x`, `p` are `t_INT`s, and `p` is assumed to be prime. Return `NULL` if `x` is not a quadratic residue modulo `p`.

`GEN Fp_sqrtn(GEN x, GEN n, GEN p, GEN *zn)` returns an `n`-th root of `x` modulo `p` (smallest non-negative residue), where `x`, `n`, `p` are `t_INT`s, and `p` is assumed to be prime. Return `NULL` if `x` is not an `n`-th power residue. Otherwise, if `zn` is non-`NULL` set it to a primitive `n`-th root of 1.

`GEN Zn_sqrt(GEN x, GEN n)` returns one of the square roots of `x` modulo `n` (possibly not prime), where `x` is a `t_INT` and `n` is either a `t_INT` or is given by its factorisation matrix. Return `NULL` if no such square root exist.

`long kross(long x, long y)` returns the Kronecker symbol $(x|y)$, i.e.-1, 0 or 1. If `y` is an odd prime, this is the Legendre symbol. (Contrary to `krouu`, `kross` also supports `y` = 0)

`long krouu(ulong x, ulong y)` returns the Kronecker symbol $(x|y)$, i.e. -1, 0 or 1. Assumes `y` is non-zero. If `y` is an odd prime, this is the Legendre symbol.

`long krois(GEN x, long y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and `long y`. As `kross` otherwise.

`long kroiu(GEN x, ulong y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and non-zero `ulong y`. As `krouu` otherwise.

`long krosi(long x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT y`. As `kross` otherwise.

`long kronecker(GEN x, GEN y)` returns the Kronecker symbol $(x|y)$ of `t_INT`s x and `y`. As `kross` otherwise.

`GEN pgener_Fp(GEN p)` returns the smallest primitive root modulo `p`, assuming `p` is prime.

`GEN pgener_Zp(GEN p)` returns the smallest primitive root modulo p^k, $k > 1$, assuming `p` is an odd prime.

`long Zp_issquare(GEN x, GEN p)` returns 1 if the `t_INT` x is a p-adic square, 0 otherwise.

`long Zn_issquare(GEN x, GEN n)` returns 1 if `t_INT` x is a square modulo `n` (possibly not prime), where n is either a `t_INT` or is given by its factorisation matrix. Return 0 otherwise.

`long Zn_ispower(GEN x, GEN n, GEN K, GEN *py)` returns 1 if `t_INT` x is a K-th power modulo `n` (possibly not prime), where n is either a `t_INT` or is given by its factorisation matrix. Return 0 otherwise. If `py` is not `NULL`, set it to y such that $y^K = x$ modulo n.

`GEN pgener_Fp_local(GEN p, GEN L)`, `L` being a vector of primes dividing $p-1$, returns the smallest integer $x > 1$ which is a generator of the ℓ-Sylow of $\mathbf{F}_p^*$ for every ℓ in `L`. In other words, $x^{(p-1)/\ell} \neq 1$ for all such ℓ. In particular, returns `pgener_Fp(p)` if `L` contains all primes dividing $p-1$. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square.

`GEN rootsof1_Fp(GEN n, GEN p)` returns a primitive n-th root modulo the prime p.

`GEN rootsof1u_Fp(ulong n, GEN p)` returns a primitive n-th root modulo the prime p.

`ulong rootsof1_Fl(ulong n, ulong p)` returns a primitive n-th root modulo the prime p.

6.2.19 Extending functions to vector inputs.

The following functions apply f to the given arguments, recursively if they are of vector / matrix type:

`GEN map_proto_G(GEN (*f)(GEN), GEN x)` For instance, if x is a `t_VEC`, return a `t_VEC` whose components are the $f(x[i])$.

`GEN map_proto_lG(long (*f)(GEN), GEN x)` As above, applying the function `stoi( f() )`.

`GEN map_proto_GL(GEN (*f)(GEN,long), GEN x, long y)`

`GEN map_proto_lGL(long (*f)(GEN,long), GEN x, long y)`

In the last function, f implements an associative binary operator, which we extend naturally to an n-ary operator f_n for any n: by convention, $f_0() = 1$, $f_1(x) = x$, and

$$f_n(x_1, \dots, x_n) = f(f_{n-1}(x_1, \dots, x_{n-1}), x_n)),$$

for $n \geq 2$.

`GEN gassoc_proto(GEN (*f)(GEN,GEN),GEN x, GEN y)` If y is not `NULL`, return $f(x, y)$. Otherwise, x must be of vector type, and we return the result of f applied to its components, computed using a divide-and-conquer algorithm. More precisely, return

$$f(f(x_1, \texttt{NULL}), f(x_2, \texttt{NULL})),$$

where x_1, x_2 are the two halves of x.

6.2.20 Miscellaneous arithmetic functions.

`ulong coreu(ulong n)`, unique squarefree integer d dividing n such that n/d is a square.

`ulong eulerphiu(ulong n)`, Euler's totient function of n.

`ulong eulerphiu_fact(GEN fa)`, Euler's totient function of the `ulong` n, where `fa` is `factoru(n)`.

`long moebiusu(ulong n)`, Moebius μ-function of n.

`GEN divisorsu(ulong n)`, returns the divisors of n in a `t_VECSMALL`, sorted by increasing order.

`long uissquarefree(ulong n)` returns 1 if `n` is square-free, and 0 otherwise.

`ulong uissquarefree_fact(GEN fa)` returns `uissquarefree(n)`, where `fa` is `factoru(n)`.

`long uposisfundamental(ulong x)` return 1 is x is a fundamental discriminant, and 0 otherwise.

`long unegisfundamental(ulong x)` return 1 is $-x$ is a fundamental discriminant, and 0 otherwise.

`int uis_357_power(ulong x, ulong *pt, ulong *mask)` as `is_357_power` for `ulong` x.

`int uis_357_powermod(ulong x, ulong *mask)` as `uis_357_power`, but only check for 3rd, 5th or 7th powers modulo $211 \times 209 \times 61 \times 203 \times 117 \times 31 \times 43 \times 71$.

`long uisprimepower(ulong n, ulong *p)` as `isprimepower`, for `ulong` n.

`int uislucaspsp(ulong n)` returns 1 if the `ulong` n fails Lucas compositeness test (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`ulong sumdigitsu(ulong n)` returns the sum of decimal digits of u.

`GEN usumdivkvec(ulong n, GEN K)` K being a `t_VECSMALL` of positive integers. Returns the vector of `sumdivk`$(n, K[i])$.

`GEN hilbertii(GEN x, GEN y, GEN p)`, returns the Hilbert symbol (x, y) at the prime p (`NULL` for the place at infinity); x and y are `t_INT`s.

`GEN sumdedekind(GEN h, GEN k)` returns the Dedekind sum associated to the `t_INT` h and k, $k > 0$.

`GEN sumdedekind_coprime(GEN h, GEN k)` as `sumdedekind`, except that h and k are assumed to be coprime `t_INT`s.

`GEN u_sumdedekind_coprime(long h, long k)` Let $k > 0$, $0 \leq h < k$, $(h, k) = 1$. Returns $[s_1, s_2]$ in a `t_VECSMALL`, such that $s(h, k) = (s_2 + ks_1)/(12k)$. Requires $\max(h + k/2, k) <$ `LONG_MAX` to avoid overflow, in particular $k \leq (2/3)$`LONG_MAX` is fine.

Chapter 7:
Level 2 kernel

These functions deal with modular arithmetic, linear algebra and polynomials where assumptions can be made about the types of the coefficients.

7.1 Naming scheme.

A function name is built in the following way: $A_1\_\ldots\_A_n fun$ for an operation *fun* with n arguments of class $A_1,\ldots, A_n$. A class name is given by a base ring followed by a number of code letters. Base rings are among

`Fl`: $\mathbf{Z}/l\mathbf{Z}$ where $l < 2^{\texttt{BITS\_IN\_LONG}}$ is not necessarily prime. Implemented using `ulong`s

`Fp`: $\mathbf{Z}/p\mathbf{Z}$ where p is a `t_INT`, not necessarily prime. Implemented as `t_INT`s z, preferably satisfying $0 \leq z < p$. More precisely, any `t_INT` can be used as an `Fp`, but reduced inputs are treated more efficiently. Outputs from `Fp`xxx routines are reduced.

`Fq`: $\mathbf{Z}[X]/(p, T(X))$, p a `t_INT`, T a `t_POL` with `Fp` coefficients or `NULL` (in which case no reduction modulo `T` is performed). Implemented as `t_POL`s z with `Fp` coefficients, $\deg(z) < \deg$ `T`, although z a `t_INT` is allowed for elements in the prime field.

`Z`: the integers $\mathbf{Z}$, implemented as `t_INT`s.

`z`: the integers $\mathbf{Z}$, implemented using (signed) `long`s.

`Q`: the rational numbers $\mathbf{Q}$, implemented as `t_INT`s and `t_FRAC`s.

`Rg`: a commutative ring, whose elements can be `gadd`-ed, `gmul`-ed, etc.

Possible letters are:

`X`: polynomial in X (`t_POL` in a fixed variable), e.g. `FpX` means $\mathbf{Z}/p\mathbf{Z}[X]$

`Y`: polynomial in $Y \neq X$. This is used to resolve ambiguities. E.g. `FpXY` means $((\mathbf{Z}/p\mathbf{Z})[X])[Y]$.

`V`: vector (`t_VEC` or `t_COL`), treated as a line vector (independently of the actual type). E.g. `ZV` means $\mathbf{Z}^k$ for some k.

`C`: vector (`t_VEC` or `t_COL`), treated as a column vector (independently of the actual type). The difference with `V` is purely semantic: if the result is a vector, it will be of type `t_COL` unless mentioned otherwise. For instance the function `ZC_add` receives two integral vectors (`t_COL` or `t_VEC`, possibly different types) of the same length and returns a `t_COL` whose entries are the sums of the input coefficients.

`M`: matrix (`t_MAT`). E.g. `QM` means a matrix with rational entries

`T`: Trees. Either a leaf or a `t_VEC` of trees.

`E`: point over an elliptic curve, represented as two-component vectors `[x,y]`, except for the represented by the one-component vector `[0]`. Not all curve models are supported.

`Q`: representative (`t_POL`) of a class in a polynomial quotient ring. E.g. an `FpXQ` belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(T(X))$, `FpXQV` means a vector of such elements, etc.

`x`, `y`, `m`, `v`, `c`, `q`: as their uppercase counterpart, but coefficient arrays are implemented using `t_VECSMALL`s, which coefficient understood as `ulong`s.

`x` and `y` (and `q`) are implemented by a `t_VECSMALL` whose first coefficient is used as a code-word and the following are the coefficients , similarly to a `t_POL`. This is known as a 'POLSMALL'.

`m` are implemented by a `t_MAT` whose components (columns) are `t_VECSMALL`s. This is known as a 'MATSMALL'.

`v` and `c` are regular `t_VECSMALL`s. Difference between the two is purely semantic.

Omitting the letter means the argument is a scalar in the base ring. Standard functions *fun* are

`add`: add

`sub`: subtract

`mul`: multiply

`sqr`: square

`div`: divide (Euclidean quotient)

`rem`: Euclidean remainder

`divrem`: return Euclidean quotient, store remainder in a pointer argument. Three special values of that pointer argument modify the default behavior: `NULL` (do not store the remainder, used to implement `div`), `ONLY_REM` (return the remainder, used to implement `rem`), `ONLY_DIVIDES` (return the quotient if the division is exact, and `NULL` otherwise).

`gcd`: GCD

`extgcd`: return GCD, store Bezout coefficients in pointer arguments

`pow`: exponentiate

`eval`: evaluation / composition

7.2 Modular arithmetic.

These routines implement univariate polynomial arithmetic and linear algebra over finite fields, in fact over finite rings of the form $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, where p is not necessarily prime and $T \in (\mathbf{Z}/p\mathbf{Z})[X]$ is possibly reducible; and finite extensions thereof. All this can be emulated with `t_INTMOD` and `t_POLMOD` coefficients and using generic routines, at a considerable loss of efficiency. Also, specialized routines are available that have no obvious generic equivalent.

7.2.1 `FpC` / `FpV`, `FpM`. A `ZV` (resp. a `ZM`) is a `t_VEC` or `t_COL` (resp. `t_MAT`) with `t_INT` coefficients. An `FpV` or `FpM`, with respect to a given `t_INT p`, is the same with `Fp` coordinates; operations are understood over $\mathbf{Z}/p\mathbf{Z}$.

7.2.1.1 Conversions.

`int Rg_is_Fp(GEN z, GEN *p)`, checks if `z` can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT` or a `t_INTMOD` whose modulus is equal to `*p`, (if `*p` not `NULL`), in that case return 1, else 0. If a modulus is found it is put in `*p`, else `*p` is left unchanged.

`int RgV_is_FpV(GEN z, GEN *p)`, `z` a `t_VEC` (resp. `t_COL`), checks if it can be mapped to a `FpV` (resp. `FpC`), by checking `Rg_is_Fp` coefficientwise.

`int RgM_is_FpM(GEN z, GEN *p)`, `z` a `t_MAT`, checks if it can be mapped to a `FpM`, by checking `RgV_is_FpV` columnwise.

`GEN Rg_to_Fp(GEN z, GEN p)`, `z` a scalar which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT`, a `t_INTMOD` whose modulus is divisible by p, a `t_FRAC` whose denominator is coprime to p, or a `t_PADIC` with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized.

`GEN padic_to_Fp(GEN x, GEN p)` special case of `Rg_to_Fp`, for a x a `t_PADIC`.

`GEN RgV_to_FpV(GEN z, GEN p)`, `z` a `t_VEC` or `t_COL`, returns the `FpV` (as a `t_VEC`) obtained by applying `Rg_to_Fp` coefficientwise.

`GEN RgC_to_FpC(GEN z, GEN p)`, `z` a `t_VEC` or `t_COL`, returns the `FpC` (as a `t_COL`) obtained by applying `Rg_to_Fp` coefficientwise.

`GEN RgM_to_FpM(GEN z, GEN p)`, `z` a `t_MAT`, returns the `FpM` obtained by applying `RgC_to_FpC` columnwise.

`GEN RgM_Fp_init(GEN z, GEN p, ulong *pp)`, given an `RgM` z, whose entries can be mapped to $\mathbf{F}_p$ (as per `Rg_to_Fp`), and a prime number p. This routine returns a normal form of z: either an `F2m` ($p = 2$), an `Flm` (p fits into an `ulong`) or an `FpM`. In the first two cases, `pp` is set to `itou`(p), and to 0 in the last.

The functions above are generally used as follow:

```
GEN add(GEN x, GEN y)
{
  GEN p = NULL;
  if (Rg_is_Fp(x, &p) && Rg_is_Fp(y, &p) && p)
  {
    x = Rg_to_Fp(x, p); y = Rg_to_Fp(y, p);
    z = Fp_add(x, y, p);
    return Fp_to_mod(z);
  }
  else return gadd(x, y);
}
```

`GEN FpC_red(GEN z, GEN p)`, `z` a `ZC`. Returns `lift(Col(z) * Mod(1,p))`, hence a `t_COL`.

`GEN FpV_red(GEN z, GEN p)`, `z` a `ZV`. Returns `lift(Vec(z) * Mod(1,p))`, hence a `t_VEC`

`GEN FpM_red(GEN z, GEN p)`, `z` a `ZM`. Returns `lift(z * Mod(1,p))`, which is an `FpM`.

7.2.1.2 Basic operations.

`GEN FpC_center(GEN z, GEN p, GEN pov2)` returns a `t_COL` whose entries are the `Fp_center` of the `gel(z,i)`.

`GEN FpM_center(GEN z, GEN p, GEN pov2)` returns a matrix whose entries are the `Fp_center` of the `gcoeff(z,i,j)`.

`GEN FpC_add(GEN x, GEN y, GEN p)` adds the `ZC` x and y and reduce modulo p to obtain an `FpC`.

`GEN FpV_add(GEN x, GEN y, GEN p)` same as `FpC_add`, returning and `FpV`.

`GEN FpC_sub(GEN x, GEN y, GEN p)` subtracts the `ZC` y to the `ZC` x and reduce modulo p to obtain an `FpC`.

`GEN FpV_sub(GEN x, GEN y, GEN p)` same as `FpC_sub`, returning and `FpV`.

`GEN FpC_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `ZC x` (seen as a column vector) by the `t_INT y` and reduce modulo `p` to obtain an `FpC`.

`GEN FpC_FpV_mul(GEN x, GEN y, GEN p)` multiplies the `ZC x` (seen as a column vector) by the `ZV y` (seen as a row vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpM_mul(GEN x, GEN y, GEN p)` multiplies the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpM_powu(GEN x, ulong n, GEN p)` computes x^n where x is a square `FpM`.

`GEN FpM_FpC_mul(GEN x, GEN y, GEN p)` multiplies the `ZM x` by the `ZC y` (seen as a column vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpC`.

`GEN FpM_FpC_mul_FpX(GEN x, GEN y, GEN p, long v)` is a memory-clean version of

```
GEN tmp = FpM_FpC_mul(x,y,p);
return RgV_to_RgX(tmp, v);
```

`GEN FpV_FpC_mul(GEN x, GEN y, GEN p)` multiplies the `ZV x` (seen as a row vector) by the `ZC y` (seen as a column vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `Fp`.

`GEN FpV_dotproduct(GEN x,GEN y,GEN p)` scalar product of x and y (assumed to have the same length).

`GEN FpV_dotsquare(GEN x, GEN p)` scalar product of x with itself. has `t_INT` entries.

7.2.1.3 `Fp`-linear algebra. The implementations are not asymptotically efficient ($O(n^3)$ standard algorithms).

`GEN FpM_deplin(GEN x, GEN p)` returns a non-trivial kernel vector, or `NULL` if none exist.

`GEN FpM_det(GEN x, GEN p)` as `det`

`GEN FpM_gauss(GEN a, GEN b, GEN p)` as `gauss`, where b is a `FpM`.

`GEN FpM_FpC_gauss(GEN a, GEN b, GEN p)` as `gauss`, where b is a `FpC`.

`GEN FpM_image(GEN x, GEN p)` as `image`

`GEN FpM_intersect(GEN x, GEN y, GEN p)` as `intersect`

`GEN FpM_inv(GEN x, GEN p)` returns a left inverse of `x` (the inverse if x is square), or `NULL` if `x` is not invertible.

`GEN FpM_FpC_invimage(GEN m, GEN v, GEN p)` given an `FpM` x and an `FpC` y, returns an x such that $Ax = y$, or `NULL` if no such vector exist.

`GEN FpM_invimage(GEN m, GEN v, GEN p)` given two `FpM` x and y, returns x such that $Ax = y$, or `NULL` if no such matrix exist.

`GEN FpM_ker(GEN x, GEN p)` as `ker`

`long FpM_rank(GEN x, GEN p)` as `rank`

`GEN FpM_indexrank(GEN x, GEN p)` as `indexrank`

`GEN FpM_suppl(GEN x, GEN p)` as `suppl`

`GEN FpM_hess(GEN x, GEN p)` upper Hessenberg form of x over $\mathbf{F}_p$.

`GEN FpM_charpoly(GEN x, GEN p)` characteristic polynomial of x.

7.2.1.4 `FqC`, `FqM` and `Fq`-linear algebra.

An `FqM` (resp. `FqC`) is a matrix (resp a `t_COL`) with `Fq` coefficients (with respect to given `T`, `p`), not necessarily reduced (i.e arbitrary `t_INT`s and `ZX`s in the same variable as `T`).

`GEN FqC_add(GEN a, GEN b, GEN T, GEN p)`

`GEN FqC_sub(GEN a, GEN b, GEN T, GEN p)`

`GEN FqC_Fq_mul(GEN a, GEN b, GEN T, GEN p)`

`GEN FqM_deplin(GEN x, GEN T, GEN p)` returns a non-trivial kernel vector, or `NULL` if none exist.

`GEN FqM_gauss(GEN a, GEN b, GEN T, GEN p)` as `gauss`, where b is a `FqM`.

`GEN FqM_FqC_gauss(GEN a, GEN b, GEN T, GEN p)` as `gauss`, where b is a `FqC`.

`GEN FqM_FqC_mul(GEN a, GEN b, GEN T, GEN p)`

`GEN FqM_ker(GEN x, GEN T, GEN p)` as `ker`

`GEN FqM_image(GEN x, GEN T, GEN p)` as `image`

`GEN FqM_inv(GEN x, GEN T, GEN p)` returns the inverse of `x`, or `NULL` if `x` is not invertible.

`GEN FqM_mul(GEN a, GEN b, GEN T, GEN p)`

`long FqM_rank(GEN x, GEN T, GEN p)` as `rank`

`GEN FqM_suppl(GEN x, GEN T, GEN p)` as `suppl`

`GEN FqM_det(GEN x, GEN T, GEN p)` as `det`

7.2.2 `Flc / Flv, Flm`. See `FpV`, `FpM` operations.

`GEN Flv_copy(GEN x)` returns a copy of `x`.

`GEN Flv_center(GEN z, ulong p, ulong ps2)`

`GEN Flm_copy(GEN x)` returns a copy of `x`.

`GEN matid_Flm(long n)` returns an `Flm` which is an $n \times n$ identity matrix.

`GEN scalar_Flm(long s, long n)` returns an `Flm` which is s times the $n \times n$ identity matrix.

`GEN Flm_center(GEN z, ulong p, ulong ps2)`

`GEN Flm_Fl_add(GEN x, ulong y, ulong p)` returns $x + y * \mathrm{Id}$ (x must be square).

`GEN Flm_Flc_mul(GEN x, GEN y, ulong p)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_Fl_mul(GEN x, ulong y, ulong p)` multiplies the `Flm x` by `y`.

`GEN Flm_neg(GEN x, ulong p)` negates the `Flm x`.

`void Flm_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the `Flm x` by `x * y`.

`GEN Flc_Fl_mul(GEN x, ulong y, ulong p)` multiplies the `Flv x` by `y`.

`void Flc_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the `Flc x` by `x * y`.

`void Flc_Fl_mul_part_inplace(GEN x, ulong y, ulong p, long l)` multiplies $x[1..l]$ by y modulo p. In place.

`GEN Flc_Fl_div(GEN x, ulong y, ulong p)` divides the `Flv x` by `y`.

`void Flc_Fl_div_inplace(GEN x, ulong y, ulong p)` replaces the `Flv x` by `x/y`.

`void Flc_lincomb1_inplace(GEN X, GEN Y, ulong v, ulong q)` sets $X \leftarrow X + vY$, where X, Y are `Flc`. Memory efficient (e.g. no-op if $v = 0$), and gerepile-safe.

`GEN Flv_add(GEN x, GEN y, ulong p)` adds two `Flv`.

`void Flv_add_inplace(GEN x, GEN y, ulong p)` replaces x by $x + y$.

`GEN Flv_sub(GEN x, GEN y, ulong p)` subtracts `y` to `x`.

`void Flv_sub_inplace(GEN x, GEN y, ulong p)` replaces x by $x - y$.

`ulong Flv_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of `x` and `y`

`ulong Flv_sum(GEN x, ulong p)` returns the sums of the components of x.

`GEN zero_Flm(long m, long n)` creates a `Flm` with `m` x `n` components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_Flm_copy(long m, long n)` creates a `Flm` with `m` x `n` components set to 0.

`GEN zero_Flv(long n)` creates a `Flv` with `n` components set to 0.

`GEN row_Flm(GEN A, long x0)` return $A[i,]$, the i-th row of the `Flm` (or `zm`) A.

`GEN Flm_mul(GEN x, GEN y, ulong p)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_powu(GEN x, ulong n, ulong p)` computes x^n where x is a square `Flm`.

`GEN Flm_charpoly(GEN x, ulong p)` return the characteristic polynomial of the square `Flm` x, as a `Flx`.

`GEN Flm_deplin(GEN x, ulong p)`

`ulong Flm_det(GEN x, ulong p)`

`ulong Flm_det_sp(GEN x, ulong p)`, as `Flm_det`, in place (destroys `x`).

`GEN Flm_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flm`.

`GEN Flm_Flc_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flc`.

`GEN Flm_indexrank(GEN x, ulong p)`

`GEN Flm_inv(GEN x, ulong p)`

`GEN Flm_Flc_invimage(GEN A, GEN y, ulong p)` given an `Flm` x and an `Flc` y, returns an x such that $Ax = y$, or `NULL` if no such vector exist.

`GEN Flm_invimage(GEN x, GEN y, ulong p)` given two `Flm` x and y, returns x such that $Ax = y$, or `NULL` if no such matrix exist.

`GEN Flm_ker(GEN x, ulong p)`

`GEN Flm_ker_sp(GEN x, ulong p, long deplin)`, as `Flm_ker` (if `deplin=0`) or `Flm_deplin` (if `deplin=1`) , in place (destroys `x`).

`long Flm_rank(GEN x, ulong p)`

`long Flm_suppl(GEN x, ulong p)`

`GEN Flm_image(GEN x, ulong p)`

`GEN Flm_transpose(GEN x)`

`GEN Flm_hess(GEN x, ulong p)` upper Hessenberg form of x over $\mathbf{F}_p$.

7.2.3 `F2c / F2v, F2m`. An `F2v v` is a `t_VECSMALL` representing a vector over $\mathbf{F}_2$. Specifically `z[0]` is the usual codeword, `z[1]` is the number of components of v and the coefficients are given by the bits of remaining words by increasing indices.

`ulong F2v_coeff(GEN x, long i)` returns the coefficient $i \geq 1$ of x.

`void F2v_clear(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 0.

`void F2v_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 1$ of x.

`void F2v_set(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 1.

`void F2v_copy(GEN x)` returns a copy of x.

`GEN F2v_slice(GEN x, long a, long b)` returns the `F2v` with entries $x[a]$, ..., $x[b]$. Assumes $a \leq b$.

`ulong F2m_coeff(GEN x, long i, long j)` returns the coefficient (i, j) of x.

`void F2m_clear(GEN x, long i, long j)` sets the coefficient (i, j) of x to 0.

`void F2m_flip(GEN x, long i, long j)` adds 1 to the coefficient (i, j) of x.

`void F2m_set(GEN x, long i, long j)` sets the coefficient (i, j) of x to 1.

`void F2m_copy(GEN x)` returns a copy of x.

`GEN F2m_rowslice(GEN x, long a, long b)` returns the `F2m` built from the a-th to b-th rows of the `F2m` x. Assumes $a \leq b$.

`GEN F2m_F2c_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN F2m_image(GEN x)` gives a subset of the columns of x that generate the image of x.

`GEN F2m_invimage(GEN A, GEN B)`

`GEN F2m_F2c_invimage(GEN A, GEN y)`

`GEN F2m_gauss(GEN a, GEN b)` as `gauss`, where b is a `F2m`.

`GEN F2m_F2c_gauss(GEN a, GEN b)` as `gauss`, where b is a `F2c`.

`GEN F2m_indexrank(GEN x)` x being a matrix of rank r, returns a vector with two `t_VECSMALL` components y and z of length r giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using `vecextract`(x, y, z) is invertible.

`GEN F2m_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN F2m_powu(GEN x, ulong n)` computes x^n where x is a square `F2m`.

`long F2m_rank(GEN x)` as `rank`.

`long F2m_suppl(GEN x)` as `suppl`.

`GEN matid_F2m(long n)` returns an `F2m` which is an $n \times n$ identity matrix.

`GEN zero_F2v(long n)` creates a `F2v` with `n` components set to 0.

`GEN F2v_ei(long n, long i)` creates a `F2v` with `n` components set to 0, but for the i-th one, which is set to 1 (i-th vector in the canonical basis).

`GEN zero_F2m(long m, long n)` creates a `Flm` with `m` x `n` components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_F2m_copy(long m, long n)` creates a `F2m` with `m` x `n` components set to 0.

`GEN F2c_to_Flc(GEN x)`

`GEN F2c_to_ZC(GEN x)`

`GEN ZV_to_F2v(GEN x)`

`GEN RgV_to_F2v(GEN x)`

`GEN F2m_to_Flm(GEN x)`

`GEN F2m_to_ZM(GEN x)`

`GEN Flv_to_F2v(GEN x)`

`GEN Flm_to_F2m(GEN x)`

`GEN ZM_to_F2m(GEN x)`

`GEN RgM_to_F2m(GEN x)`

`void F2v_add_inplace(GEN x, GEN y)` replaces x by $x + y$. It is allowed for y to be shorter than x.

`ulong F2m_det(GEN x)`

`ulong F2m_det_sp(GEN x)`, as `F2m_det`, in place (destroys `x`).

`GEN F2m_deplin(GEN x)`

`ulong F2v_dotproduct(GEN x, GEN y)` returns the scalar product of `x` and `y`

`GEN F2m_inv(GEN x)`

`GEN F2m_ker(GEN x)`

`GEN F2m_ker_sp(GEN x, long deplin)`, as `F2m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys `x`).

7.2.4 `FlxqV`, `FlxqM`. See `FqV`, `FqM` operations.

`GEN FlxqV_dotproduct(GEN x, GEN y, GEN T, ulong p)` as `FpV_dotproduct`.

`GEN FlxM_Flx_add_shallow(GEN x, GEN y, ulong p)` as `RgM_Rg_add_shallow`.

`GEN FlxqM_gauss(GEN a, GEN b, GEN T, ulong p)`

`GEN FlxqM_FlxqC_gauss(GEN a, GEN b, GEN T, ulong p)`

`GEN FlxqM_FlxqC_mul(GEN a, GEN b, GEN T, ulong p)`

`GEN FlxqM_ker(GEN x, GEN T, ulong p)`

`GEN FlxqM_image(GEN x, GEN T, ulong p)`

`GEN FlxqM_det(GEN a, GEN T, ulong p)`

`GEN FlxqM_inv(GEN x, GEN T, ulong p)`

`GEN FlxqM_mul(GEN a, GEN b, GEN T, ulong p)`

`long FlxqM_rank(GEN x, GEN T, ulong p)`

`GEN matid_FlxqM(long n, GEN T, ulong p)`

7.2.5 `Zlm`. `GEN ZlM_gauss(GEN a, GEN b, ulong p, long e, GEN C)` as `gauss` with the following peculiarities: a and b are `ZM`, such that a is invertible modulo p. Optional C is an `Flm` that is an inverse of $a \bmod p$ or `NULL`. Return the matrix x such that $ax = b \bmod p^e$ and all elements of x are in $[0, p^e - 1]$. For efficiency, it is better to reduce a and b mod p^e first.

7.2.6 `FpX`. Let `p` an understood `t_INT`, to be given in the function arguments; in practice `p` is not assumed to be prime, but be wary. Recall than an `Fp` object is a `t_INT`, preferably belonging to $[0, \mathtt{p} - 1]$; an `FpX` is a `t_POL` in a fixed variable whose coefficients are `Fp` objects. Unless mentioned otherwise, all outputs in this section are `FpX`s. All operations are understood to take place in $(\mathbf{Z}/\mathrm{p}\mathbf{Z})[X]$.

7.2.6.1 Conversions. In what follows p is always a t_INT, not necessarily prime.

int RgX_is_FpX(GEN z, GEN *p), z a t_POL, checks if it can be mapped to a FpX, by checking Rg_is_Fp coefficientwise.

GEN RgX_to_FpX(GEN z, GEN p), z a t_POL, returns the FpX obtained by applying Rg_to_Fp coefficientwise.

GEN FpX_red(GEN z, GEN p), z a ZX, returns lift(z * Mod(1,p)), normalized.

GEN FpXV_red(GEN z, GEN p), z a t_VEC of ZX. Applies FpX_red componentwise and returns the result (and we obtain a vector of FpXs).

GEN FpXT_red(GEN z, GEN p), z a tree of ZX. Applies FpX_red to each leaf and returns the result (and we obtain a tree of FpXs).

7.2.6.2 Basic operations. In what follows p is always a t_INT, not necessarily prime.

Now, except for p, the operands and outputs are all FpX objects. Results are undefined on other inputs.

GEN FpX_add(GEN x,GEN y, GEN p) adds x and y.

GEN FpX_neg(GEN x,GEN p) returns $-$x, the components are between 0 and p if this is the case for the components of x.

GEN FpX_renormalize(GEN x, long l), as normalizepol, where l = lg(x), in place.

GEN FpX_sub(GEN x,GEN y,GEN p) returns $x - y$.

GEN FpX_mul(GEN x,GEN y,GEN p) returns xy.

GEN FpX_mulspec(GEN a, GEN b, GEN p, long na, long nb) see ZX_mulspec

GEN FpX_sqr(GEN x,GEN p) returns x^2.

GEN FpX_divrem(GEN x, GEN y, GEN p, GEN *pr) returns the quotient of x by y, and sets pr to the remainder.

GEN FpX_div(GEN x, GEN y, GEN p) returns the quotient of x by y.

GEN FpX_div_by_X_x(GEN A, GEN a, GEN p, GEN *r) returns the quotient of the FpX A by $(X -$ a), and sets r to the remainder A(a).

GEN FpX_rem(GEN x, GEN y, GEN p) returns the remainder x mod y.

long FpX_valrem(GEN x, GEN t, GEN p, GEN *r) The arguments x and e being non-zero FpX returns the highest exponent e such that t^e divides x. The quotient x/t^e is returned in *r. In particular, if t is irreducible, this returns the valuation at t of x, and *r is the prime-to-t part of x.

GEN FpX_deriv(GEN x, GEN p) returns the derivative of x. This function is not memory-clean, but nevertheless suitable for gerepileupto.

GEN FpX_translate(GEN P, GEN c, GEN p) let c be an Fp and let P be an FpX; returns the translated FpX of $P(X + c)$.

GEN FpX_gcd(GEN x, GEN y, GEN p) returns a (not necessarily monic) greatest common divisor of x and y.

GEN FpX_halfgcd(GEN x, GEN y, GEN p) returns a two-by-two FpXM M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN FpX_extgcd(GEN x, GEN y, GEN p, GEN *u, GEN *v) returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$ (not necessarily monic), and sets *u, *v to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$. If *u is set to NULL, it is not computed which is a bit faster. This is useful when computing the inverse of y modulo x.

GEN FpX_center(GEN z, GEN p, GEN pov2) returns the polynomial whose coefficient belong to the symmetric residue system. Assumes the coefficients already belong to $[0, \mathtt{p} - 1]$) and pov2 is shifti(p,-1).

7.2.6.3 Mixed operations. The following functions implement arithmetic operations between FpX and Fp operands, the result being of type FpX. The integer p need not be prime.

GEN Z_to_FpX(GEN x, GEN p, long v) converts a t_INT to a scalar polynomial in variable v, reduced modulo p.

GEN FpX_Fp_add(GEN y, GEN x, GEN p) add the Fp x to the FpX y.

GEN FpX_Fp_add_shallow(GEN y, GEN x, GEN p) add the Fp x to the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN FpX_Fp_sub(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y.

GEN FpX_Fp_sub_shallow(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN Fp_FpX_sub(GEN x,GEN y,GEN p) returns $x - y$, where x is a t_INT and y an FpX.

GEN FpX_Fp_mul(GEN x, GEN y, GEN p) multiplies the FpX x by the Fp y.

GEN FpX_Fp_mulspec(GEN x, GEN y, GEN p, long lx) see ZX_mulspec

GEN FpX_mulu(GEN x, ulong y, GEN p) multiplies the FpX x by y.

GEN FpX_Fp_mul_to_monic(GEN y,GEN x,GEN p) returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

7.2.6.4 Miscellaneous operations.

GEN FpX_normalize(GEN z, GEN p) divides the FpX z by its leading coefficient. If the latter is 1, z itself is returned, not a copy. If not, the inverse remains uncollected on the stack.

GEN FpX_invBarrett(GEN T, GEN p), returns the Barrett inverse M of T defined by $M(x)x^n \times T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T.

GEN FpX_rescale(GEN P, GEN h, GEN p) returns $h^{\deg(P)}P(x/h)$. P is an FpX and h is a non-zero Fp (the routine would work with any non-zero t_INT but is not efficient in this case).

GEN FpX_eval(GEN x, GEN y, GEN p) evaluates the FpX x at the Fp y. The result is an Fp.

GEN FpXV_FpC_mul(GEN V, GEN W, GEN p) multiplies a non-empty line vector ofFpX by a column vector of Fp of compatible dimensions. The result is an FpX.

GEN FpXV_prod(GEN V, GEN p), V being a vector of FpX, returns their product.

GEN FpV_roots_to_pol(GEN V, GEN p, long v), V being a vector of INTs, returns the monic FpX $\prod_i (\mathtt{pol\_x}[\mathtt{v}] - \mathtt{V}[\mathtt{i}])$.

GEN FpX_chinese_coprime(GEN x,GEN y, GEN Tx,GEN Ty, GEN Tz, GEN p): returns an FpX, congruent to x mod Tx and to y mod Ty. Assumes Tx and Ty are coprime, and Tz = Tx * Ty or NULL (in which case it is computed within).

`GEN FpV_polint(GEN x, GEN y, GEN p)` returns the `FpX` interpolation polynomial with value `y[i]` at `x[i]`. Assumes lengths are the same, components are `t_INT`s, and the `x[i]` are distinct modulo `p`.

`int FpX_is_squarefree(GEN f, GEN p)` returns 1 if the `FpX f` is squarefree, 0 otherwise.

`int FpX_is_irred(GEN f, GEN p)` returns 1 if the `FpX f` is irreducible, 0 otherwise. Assumes that `p` is prime. If `f` has few factors, `FpX_nbfact(f,p) == 1` is much faster.

`int FpX_is_totally_split(GEN f, GEN p)` returns 1 if the `FpX f` splits into a product of distinct linear factors, 0 otherwise. Assumes that `p` is prime.

`GEN FpX_factor(GEN f, GEN p)`, factors the `FpX f`. Assumes that `p` is prime. The returned value `v` is a `t_VEC` with two components: `v[1]` is a vector of distinct irreducible (`FpX`) factors, and `v[2]` is a `t_VECSMALL` of corresponding exponents. The order of the factors is deterministic (the computation is not).

`long FpX_nbfact(GEN f, GEN p)`, assuming the `FpX` f is squarefree, returns the number of its irreducible factors. Assumes that `p` is prime.

`long FpX_degfact(GEN f, GEN p)`, as `FpX_factor`, but the degrees of the irreducible factors are returned instead of the factors themselves (as a `t_VECSMALL`). Assumes that `p` is prime.

`long FpX_nbroots(GEN f, GEN p)` returns the number of distinct roots in $\mathbf{Z}/p\mathbf{Z}$ of the `FpX f`. Assumes that `p` is prime.

`GEN FpX_oneroot(GEN f, GEN p)` returns one root in $\mathbf{Z}/p\mathbf{Z}$ of the `FpX f`. Return `NULL` if no root exists. Assumes that `p` is prime.

`GEN FpX_roots(GEN f, GEN p)` returns the roots in $\mathbf{Z}/p\mathbf{Z}$ of the `FpX f` (without multiplicity, as a vector of `Fp`s). Assumes that `p` is prime.

`GEN random_FpX(long d, long v, GEN p)` returns a random `FpX` in variable `v`, of degree less than `d`.

`GEN FpX_resultant(GEN x, GEN y, GEN p)` returns the resultant of `x` and `y`, both `FpX`. The result is a `t_INT` belonging to $[0, p-1]$.

`GEN FpX_disc(GEN x, GEN p)` returns the discriminant of the `FpX x`. The result is a `t_INT` belonging to $[0, p-1]$.

`GEN FpX_FpXY_resultant(GEN a, GEN b, GEN p)`, `a` a `t_POL` of `t_INT`s (say in variable X), `b` a `t_POL` (say in variable X) whose coefficients are either `t_POL`s in $\mathbf{Z}[Y]$ or `t_INT`s. Returns $\mathrm{Res}_X(a, b)$ in $\mathbf{F}_p[Y]$ as an `FpY`. The function assumes that X has lower priority than Y.

7.2.7 `FpXQ, Fq`. Let `p` a `t_INT` and `T` an `FpX` for `p`, both to be given in the function arguments; an `FpXQ` object is an `FpX` whose degree is strictly less than the degree of `T`. An `Fq` is either an `FpXQ` or an `Fp`. Both represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, in which all operations below take place. In addition, `Fq` routines also allow `T` = `NULL`, in which case no reduction mod `T` is performed on the result.

For efficiency, the routines in this section may leave small unused objects behind on the stack (their output is still suitable for `gerepileupto`). Besides `T` and `p`, arguments are either `FpXQ` or `Fq` depending on the function name. (All `Fq` routines accept `FpXQ`s by definition, not the other way round.)

7.2.7.1 Preconditioned reduction.

For faster reduction, the modulus `T` can be replaced by an extended modulus, which is an `FpXT`, in all `FpXQ`- and `Fq`-classes functions, and in `FpX_rem` and `FpX_divrem`.

`GEN FpX_get_red(GEN T, GEN p)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FpX_mod(GEN eT)` returns the underlying modulus `T`.

`GEN get_FpX_var(GEN eT)` returns the variable number of the modulus.

`GEN get_FpX_degree(GEN eT)` returns the degree of the modulus.

Furthermore, `ZXT_to_FlxT` allows to convert an extended modulus for a `FpX` to an extended modulus for the corresponding `Flx`.

7.2.7.2 Conversions.

`GEN Rg_is_FpXQ(GEN z, GEN *T, GEN *p)`, checks if `z` is a `GEN` which can be mapped to $\mathbf{F}_p[X]/(T)$: anything for which `Rg_is_Fp` return 1, a `t_POL` for which `RgX_to_FpX` return 1, a `t_POLMOD` whose modulus is equal to `*T` if `*T` is not `NULL` (once mapped to a `FpX`). If an integer modulus is found it is put in `*p`, else `*p` is left unchanged. If a polynomial modulus is found it is put in `*T`, else `*T` is left unchanged.

`int RgX_is_FpXQX(GEN z, GEN *T, GEN *p)`, `z` a `t_POL`, checks if it can be mapped to a `FpXQX`, by checking `Rg_is_FpXQ` coefficientwise.

`GEN Rg_to_FpXQ(GEN z, GEN T, GEN p)`, `z` a `GEN` which can be mapped to $\mathbf{F}_p[X]/(T)$: anything `Rg_to_Fp` can be applied to, a `t_POL` to which `RgX_to_FpX` can be applied to, a `t_POLMOD` whose modulus is divisible by T (once mapped to a `FpX`), a suitable `t_RFRAC`. Returns `z` as an `FpXQ`, normalized.

`GEN RgX_to_FpXQX(GEN z, GEN T, GEN p)`, `z` a `t_POL`, returns the `FpXQ` obtained by applying `Rg_to_FpXQ` coefficientwise.

`GEN RgX_to_FqX(GEN z, GEN T, GEN p)`: let `z` be a `t_POL`; returns the `FpXQ` obtained by applying `Rg_to_FpXQ` coefficientwise and simplifying scalars to `t_INT`s.

`GEN Fq_red(GEN x, GEN T, GEN p)`, `x` a `ZX` or `t_INT`, reduce it to an `Fq` (`T` = `NULL` is allowed iff `x` is a `t_INT`).

`GEN FqX_red(GEN x, GEN T, GEN p)`, `x` a `t_POL` whose coefficients are `ZX`s or `t_INT`s, reduce them to `Fq`s. (If `T` = `NULL`, as `FpXX_red(x, p)`.)

`GEN FqV_red(GEN x, GEN T, GEN p)`, `x` a vector of `ZX`s or `t_INT`s, reduce them to `Fq`s. (If `T` = `NULL`, only reduce components mod `p` to `FpX`s or `Fp`s.)

`GEN FpXQ_red(GEN x, GEN T,GEN p)` `x` a `t_POL` whose coefficients are `t_INT`s, reduce them to `FpXQ`s.

7.2.8 FpXQ.

GEN FpXQ_add(GEN x, GEN y, GEN T,GEN p)

GEN FpXQ_sub(GEN x, GEN y, GEN T,GEN p)

GEN FpXQ_mul(GEN x, GEN y, GEN T,GEN p)

GEN FpXQ_sqr(GEN x, GEN T, GEN p)

GEN FpXQ_div(GEN x, GEN y, GEN T,GEN p)

GEN FpXQ_inv(GEN x, GEN T, GEN p) computes the inverse of x

GEN FpXQ_invsafe(GEN x,GEN T,GEN p), as FpXQ_inv, returning NULL if x is not invertible.

GEN FpXQX_renormalize(GEN x, long lx)

GEN FpXQ_pow(GEN x, GEN n, GEN T, GEN p) computes x^n.

GEN FpXQ_powu(GEN x, ulong n, GEN T, GEN p) computes x^n for small n.

GEN FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) Let g be of order ord in the finite field $\mathbf{F}_p[X]/(T)$, return e such that $a^e = g$. If e does not exists, the result is currently undefined. Assumes that T is irreducible mod p.

GEN Fp_FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) As FpXQ_log, a being a Fp.

GEN FpXQ_order(GEN a, GEN ord, GEN T, GEN p) returns the order of the FpXQ a. If o is non-NULL, it is assumed that o is a multiple of the order of a, either as a t_INT or a factorization matrix. Assumes that T is irreducible mod p.

int FpXQ_issquare(GEN x, GEN T, GEN p) returns 1 if x is a square and 0 otherwise. Assumes that T is irreducible mod p.

GEN FpXQ_sqrt(GEN x, GEN T, GEN p) returns a square root of x. Return NULL if x is not a square.

GEN FpXQ_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN *zn) returns an n-th root of x. Return NULL if x is not an n-th power residue. Otherwise, if zn is non-NULL set it to a primitive n-th root of the unity. Assumes that T is irreducible mod p.

7.2.9 Fq.

GEN Fq_add(GEN x, GEN y, GEN T/*unused*/, GEN p)

GEN Fq_sub(GEN x, GEN y, GEN T/*unused*/, GEN p)

GEN Fq_mul(GEN x, GEN y, GEN T, GEN p)

GEN Fq_Fp_mul(GEN x, GEN y, GEN T, GEN p) multiplies the Fq x by the t_INT y.

GEN Fq_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the Fq x by the scalar y.

GEN Fq_sqr(GEN x, GEN T, GEN p)

GEN Fq_neg(GEN x, GEN T, GEN p)

GEN Fq_neg_inv(GEN x, GEN T, GEN p) computes $-x^{-1}$

GEN Fq_inv(GEN x, GEN pol, GEN p) computes x^{-1}, raising an error if x is not invertible.

GEN Fq_invsafe(GEN x, GEN pol, GEN p) as Fq_inv, but returns NULL if x is not invertible.

GEN Fq_div(GEN x, GEN y, GEN T, GEN p)

GEN FqV_inv(GEN x, GEN T, GEN p) x being a vector of Fqs, return the vector of inverses of the $x[i]$. The routine uses Montgomery's trick, and involves a single inversion, plus $3(N-1)$ multiplications for N entries. The routine is not stack-clean: $2N$ FpXQ are left on stack, besides the N in the result.

GEN Fq_pow(GEN x, GEN n, GEN pol, GEN p) returns $\mathtt{x}^{\mathtt{n}}$.

GEN Fq_powu(GEN x, ulong n, GEN pol, GEN p) returns $\mathtt{x}^{\mathtt{n}}$ for small n.

int Fq_issquare(GEN x, GEN T, GEN p) returns 1 if x is a square and 0 otherwise. Assumes that T is irreducible mod p. $T = \mathtt{NULL}$ is forbidden unless x is an Fp.

GEN Fq_sqrt(GEN x, GEN T, GEN p) returns a square root of x. Return NULL if x is not a square.

GEN Fq_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN *zn) returns an n-th root of x. Return NULL if x is not an n-th power residue. Otherwise, if zn is non-NULL set it to a primitive n-th root of the unity. Assumes that T is irreducible mod p.

GEN FpXQ_charpoly(GEN x, GEN T, GEN p) returns the characteristic polynomial of x

GEN FpXQ_minpoly(GEN x, GEN T, GEN p) returns the minimal polynomial of x

GEN FpXQ_norm(GEN x, GEN T, GEN p) returns the norm of x

GEN FpXQ_trace(GEN x, GEN T, GEN p) returns the trace of x

GEN FpXQ_conjvec(GEN x, GEN T, GEN p) returns the vector of conjugates $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$ where n is the degree of T.

GEN gener_FpXQ(GEN T, GEN p, GEN *po) returns a primitive root modulo (T, p). T is an FpX assumed to be irreducible modulo the prime p. If po is not NULL it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

GEN gener_FpXQ_local(GEN T, GEN p, GEN L), L being a vector of primes dividing $p^{\deg T} - 1$, returns an element of $G := \mathbf{F}_p[x]/(T)$ which is a generator of the ℓ-Sylow of G for every ℓ in L. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square if p is odd.

GEN FpXQ_powers(GEN x, long n, GEN T, GEN p) returns $[\mathtt{x}^0, \dots, \mathtt{x}^{\mathtt{n}}]$ as a t_VEC of FpXQs.

GEN FpXQ_matrix_pow(GEN x, long m, long n, GEN T, GEN p), as FpXQ_powers$(x, n-1, T, p)$, but returns the powers as a an $m \times n$ matrix. Usually, we have $m = n = \deg T$.

GEN FpXQ_autpow(GEN a, ulong n, GEN T, GEN p) computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_p[X]/T(X)$.

GEN FpXQ_autsum(GEN a, ulong n, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FpXQ_autpowers(GEN S, long n, GEN T, GEN p) returns $[x, S(x), S(S(x)), \dots, S^{(n)}(x)]$ as a t_VEC of FpXQs.

GEN FpX_FpXQ_eval(GEN f,GEN x,GEN T,GEN p) returns f(x).

GEN FpX_FpXQV_eval(GEN f,GEN V,GEN T,GEN p) returns f(x), assuming that V was computed by FpXQ_powers(x, n, T, p).

7.2.10 `FpXX`, `FpXY`. Contrary to what the name implies, an `FpXX` is a `t_POL` whose coefficients are either `t_INT`s or `FpX`s. This reduces memory overhead at the expense of consistency. The prefix `FpXY` is an alias for `FpXX` when variables matters.

`GEN FpXX_red(GEN z, GEN p)`, `z` a `t_POL` whose coefficients are either `ZX`s or `t_INT`s. Returns the `t_POL` equal to `z` with all components reduced modulo `p`.

`GEN FpXX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`GEN FpXX_add(GEN x, GEN y, GEN p)` adds `x` and `y`.

`GEN FpXX_sub(GEN x, GEN y, GEN p)` returns `x` − `y`.

`GEN FpXX_neg(GEN x, GEN p)` returns −`x`.

`GEN FpXX_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `FpXX` `x` by the `Fp` `y`.

`GEN FpXX_FpX_mul(GEN x, GEN y, GEN p)` multiplies the coefficients of the `FpXX` `x` by the `FpX` `y`.

`GEN FpXX_mulu(GEN x, GEN y, GEN p)` multiplies the `FpXX` `x` by the scalar `y`.

`GEN FpXY_eval(GEN Q, GEN y, GEN x, GEN p)` Q being an `FpXY`, i.e. a `t_POL` with `Fp` or `FpX` coefficients representing an element of $\mathbf{F}_p[X][Y]$. Returns the `Fp` $Q(x, y)$.

`GEN FpXY_evalx(GEN Q, GEN x, GEN p)` Q being an `FpXY`, returns the `FpX` $Q(x, Y)$, where Y is the main variable of Q.

`GEN FpXY_evaly(GEN Q, GEN y, GEN p, long vx)` Q an `FpXY`, returns the `FpX` $Q(X, y)$, where X is the second variable of Q, and `vx` is the variable number of X.

`GEN FpXY_Fq_evaly(GEN Q, GEN y, GEN T, GEN p, long vx)` Q an `FpXY` and y being an `Fq`, returns the `FqX` $Q(X, y)$, where X is the second variable of Q, and `vx` is the variable number of X.

`GEN FpXY_FpXQ_evalx(GEN x, GEN Y, ulong p)` Q an `FpXY` and y being an `FpXQ`, returns the `FpXQX` $Q(x, Y)$, where Y is the first variable of Q.

`GEN FpXYQQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p)`, `x` being a `FpXY`, `T` being a `FpX` and `S` being a `FpY`, return $x^n \pmod{S, T, p}$.

7.2.11 `FpXQX`, `FqX`. Contrary to what the name implies, an `FpXQX` is a `t_POL` whose coefficients are `Fq`s. So the only difference between `FqX` and `FpXQX` routines is that `T` = `NULL` is not allowed in the latter. (It was thought more useful to allow `t_INT` components than to enforce strict consistency, which would not imply any efficiency gain.)

7.2.11.1 Basic operations.

`GEN FqX_add(GEN x,GEN y,GEN T,GEN p)`

`GEN FqX_Fq_add(GEN x, GEN y, GEN T, GEN p)` adds the `Fq` `y` to the `FqX` `x`.

`GEN FqX_neg(GEN x,GEN T,GEN p)`

`GEN FqX_sub(GEN x,GEN y,GEN T,GEN p)`

`GEN FqX_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN FqX_Fq_mul(GEN x, GEN y, GEN T, GEN p)` multiplies the `FqX` `x` by the `Fq` `y`.

`GEN FqX_mulu(GEN x, ulong y, GEN T, GEN p)` multiplies the `FqX` `x` by the scalar `y`.

`GEN FqX_Fp_mul(GEN x, GEN y, GEN T, GEN p)` multiplies the `FqX` `x` by the `t_INT` `y`.

`GEN FqX_Fq_mul_to_monic(GEN x, GEN y, GEN T, GEN p)` returns xy assuming the result is monic of the same degree as x (in particular $y \neq 0$).

`GEN FqX_normalize(GEN z, GEN T, GEN p)` divides the `FqX z` by its leading term. The leading coefficient becomes 1 as a `t_INT`.

`GEN FqX_sqr(GEN x, GEN T, GEN p)`

`GEN FqX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *z)`

`GEN FqX_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FqX_rem(GEN x, GEN y, GEN T, GEN p)`

`GEN FqX_deriv(GEN x, GEN T, GEN p)` returns the derivative of `x`. (This function is suitable for `gerepilupto` but not memory-clean.)

`GEN FqX_translate(GEN P, GEN c, GEN T, GEN p)` let c be an `Fq` defined modulo (p, T), and let P be an `FqX`; returns the translated `FqX` of $P(X + c)$.

`GEN FqX_gcd(GEN P, GEN Q, GEN T, GEN p)` returns a (not necessarily monic) greatest common divisor of x and y.

`GEN FqX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)` returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$ (not necessarily monic), and sets `*u`, `*v` to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$.

`GEN FqX_eval(GEN x, GEN y, GEN T, GEN p)` evaluates the `FqX x` at the `Fq y`. The result is an `Fq`.

`GEN FqXY_eval(GEN Q, GEN y, GEN x, GEN T, GEN p)` Q an `FqXY`, i.e. a `t_POL` with `Fq` or `FqX` coefficients representing an element of $\mathbf{F}_q[X][Y]$. Returns the `Fq` $Q(x, y)$.

`GEN FqXY_evalx(GEN Q, GEN x, GEN T, GEN p)` Q being an `FqXY`, returns the `FqX` $Q(x, Y)$, where Y is the main variable of Q.

`GEN FpXQX_red(GEN z, GEN T, GEN p)` `z` a `t_POL` whose coefficients are `ZX`s or `t_INT`s, reduce them to `FpXQ`s.

`GEN FpXQX_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN Kronecker_to_FpXQX(GEN z, GEN T, GEN p)`. Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t)$ mod $(p, T(X))$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN FpXQX_FpXQ_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_sqr(GEN x, GEN T, GEN p)`

`GEN FpXQX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *pr)`

`GEN FpXQX_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_rem(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_invBarrett(GEN y, GEN T, GEN p)` returns the Barrett inverse of the `FpXQX` y, namely a lift of $1/\mathtt{polrecip}(y) + O(x^{\deg(y)-1})$.

`GEN FpXQX_rem_Barrett(GEN x, GEN iy, GEN y, GEN T, GEN p)` returns x mod y, assuming `iy` is the Barrett inverse of y.

`GEN FpXQX_divrem_Barrett(GEN x, GEN iy, GEN y, GEN T, GEN p, GEN *pr)` performs the Euclidean division of x by y, assuming `iy` is the Barrett inverse of y. Returns the quotient and set `*pr` to the remainder.

`GEN FpXQXV_prod(GEN V, GEN T, GEN p)`, `V` being a vector of `FpXQX`, returns their product.

`GEN FpXQX_gcd(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)`

`GEN FpXQX_FpXQXQ_eval(GEN f,GEN x,GEN S, GEN T,GEN p)` returns `f(x)`.

`GEN FpXQX_FpXQXQV_eval(GEN f,GEN V,GEN S,GEN T,GEN p)` returns `f(x)`, assuming that `V` was computed by `FpXQXQ_powers`$(\mathtt{x}, n, \mathtt{S}, \mathtt{T}, \mathtt{p})$.

`GEN FpXQXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FpXQX`s, returns $\mathtt{x}*\mathtt{y}^{-1}$ modulo `S`.

`GEN FpXQXQ_inv(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $\mathtt{x}^{-1}$ modulo `S`.

`GEN FpXQXQ_invsafe(GEN x, GEN S, GEN T,GEN p)`, as `FpXQXQ_inv`, returning `NULL` if `x` is not invertible.

`GEN FpXQXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FpXQX`s, returns `xy` modulo `S`.

`GEN FpXQXQ_sqr(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $\mathtt{x}^2$ modulo `S`.

`GEN FpXQXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $\mathtt{x}^{\mathtt{n}}$ modulo `S`.

`GEN FpXQXQ_powers(GEN x, long n, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{n}}]$ as a `t_VEC` of `FpXQXQ`s.

`GEN FpXQXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p)` returns the same powers of `x` as `FpXQXQ_powers`$(x, n-1, S, T, p)$, but as an $m \times n$ matrix.

`GEN FpXQXQV_autpow(GEN a, long n, GEN S, GEN T, GEN p)` σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$, returns $[\sigma^n(X), \sigma^n(Y)]$.

`GEN FpXQXQV_autsum(GEN a, long n, GEN S, GEN T, GEN p)` σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b\sigma(b) \ldots \sigma^{n-1}(b)]$ where $b = a[3]$.

`GEN FqXQ_add(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FqX`s, returns `x+y` modulo `S`.

`GEN FqXQ_sub(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FqX`s, returns `x-y` modulo `S`.

`GEN FqXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FqX`s, returns `xy` modulo `S`.

`GEN FqXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns `x/y` modulo `S`.

`GEN FqXQ_inv(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns $\mathtt{x}^{-1}$ modulo `S`.

GEN FqXQ_invsafe(GEN x, GEN S, GEN T, GEN p) , as FqXQ_inv, returning NULL if x is not invertible.

GEN FqXQ_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns x^2 modulo S.

GEN FqXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FqXs, returns x^n modulo S.

GEN FqXQ_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FqXs, returns $[x^0, \dots, x^n]$ as a t_VEC of FqXQs.

GEN FqXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FqXQ_powers$(x, n-1, S, T, p)$, but as an $m \times n$ matrix.

GEN FqV_roots_to_pol(GEN V, GEN T, GEN p, long v), V being a vector of Fqs, returns the monic FqX $\prod_i$(pol_x[v] − V[i]).

7.2.11.2 Miscellaneous operations.

GEN init_Fq(GEN p, long n, long v) returns an irreducible polynomial of degree n > 0 over $\mathbf{F}_p$, in variable v.

int FqX_is_squarefree(GEN P, GEN T, GEN p)

GEN FqX_roots(GEN x, GEN T, GEN p) return the roots of x in $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FqX_factor(GEN x, GEN T, GEN p) same output convention as FpX_factor. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FpX_factorff(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in $\mathbf{F}_p[X]$. Factor the FpX P over the finite field $\mathbf{F}_p[Y]/(T(Y))$. See FpX_factorff_irred if P is known to be irreducible of $\mathbf{F}_p$.

GEN FpX_rootsff(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in $\mathbf{F}_p[X]$. Returns the roots of the FpX P belonging to the finite field $\mathbf{F}_p[Y]/(T(Y))$.

GEN FpX_factorff_irred(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in $\mathbf{F}_p[X]$. Factors the *irreducible* FpX P over the finite field $\mathbf{F}_p[Y]/(T(Y))$ and returns the vector of irreducible FqXs factors (the exponents, being all equal to 1, are not included).

GEN FpX_ffisom(GEN P, GEN Q, GEN p). Assumes p prime, P, Q are ZXs, both irreducible mod p, and $\deg(P) \mid \deg Q$. Outputs a monomorphism between $\mathbf{F}_p[X]/(P)$ and $\mathbf{F}_p[X]/(Q)$, as a polynomial R such that Q | P(R) in $\mathbf{F}_p[X]$. If P and Q have the same degree, it is of course an isomorphism.

void FpX_ffintersect(GEN P, GEN Q, long n, GEN p, GEN *SP,GEN *SQ, GEN MA,GEN MB) Assumes p is prime, P, Q are ZXs, both irreducible mod p, and n divides both the degree of P and Q. Compute SP and SQ such that the subfield of $\mathbf{F}_p[X]/(P)$ generated by SP and the subfield of $\mathbf{F}_p[X]/(Q)$ generated by SQ are isomorphic of degree n. The polynomials P and Q do not need to be of the same variable. If MA (resp. MB) is not NULL, it must be the matrix of the Frobenius map in $\mathbf{F}_p[X]/(P)$ (resp. $\mathbf{F}_p[X]/(Q)$).

GEN FpXQ_ffisom_inv(GEN S, GEN T, GEN p). Assumes p is prime, T a ZX, which is irreducible modulo p, S a ZX representing an automorphism of $\mathbf{F}_q := \mathbf{F}_p[X]/$(T). (S$(X)$ is the image of X by the automorphism.) Returns the inverse automorphism of S, in the same format, i.e. an FpX H such that H(S) $\equiv X$ modulo (T, p).

long FpXQX_nbfact(GEN S, GEN T, GEN p) returns the number of irreducible factors of the polynomial S over the finite field $\mathbf{F}_q$ defined by T and p.

`long FqX_nbfact(GEN S, GEN T, GEN p)` as above but accept `T=NULL`.

`long FpXQX_nbroots(GEN S, GEN T, GEN p)` returns the number of roots of the polynomial S over the finite field $\mathbf{F}_q$ defined by T and p.

`long FqX_nbroots(GEN S, GEN T, GEN p)` as above but accept `T=NULL`.

`GEN FpXQX_Frobenius(GEN S, GEN T, GEN p)` returns $X^q \pmod{S(X)}$ over the finite field $\mathbf{F}_q$ defined by T and p, thus $q = p^n$ where n is the degree of T.

`GEN FpXQXQ_halfFrobenius(GEN A, GEN S, GEN T, GEN p)` returns $A(X)^{(q-1)/2} \pmod{S(X)}$ over the finite field $\mathbf{F}_q$ defined by T and p, thus $q = p^n$ where n is the degree of T.

7.2.12 `Flx`. Let `p` an understood `ulong`, assumed to be prime, to be given the the function arguments; an `Fl` is an `ulong` belonging to $[0, \mathtt{p} - 1]$, an `Flx z` is a `t_VECSMALL` representing a polynomial with small integer coefficients. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn(`v`)` for some variable v, then the coefficients by increasing degree. An `FlxX` is a `t_POL` whose coefficients are `Flx`s.

In the following, an argument called `sv` is of the form `evalvarn`(v) for some variable number v.

7.2.12.1 Preconditioned reduction.

For faster reduction, the modulus `T` can be replaced by an extended modulus, which is an `FlxT`, in all `Flxq`-classes functions, and in `Flx_divrem`.

`GEN Flx_get_red(GEN T, ulong p)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_Flx_mod(GEN eT)` returns the underlying modulus `T`.

`GEN get_Flx_var(GEN eT)` returns the variable number of the modulus.

`GEN get_Flx_degree(GEN eT)` returns the degree of the modulus.

Furthermore, `ZXT_to_FlxT` allows to convert an extended modulus for a `FpX` to an extended modulus for the corresponding `Flx`.

7.2.12.2 Basic operations.

`ulong Flx_lead(GEN x)` returns the leading coefficient of x as a `ulong` (return 0 for the zero polynomial).

`GEN Flx_red(GEN z, ulong p)` converts from `zx` with non-negative coefficients to `Flx` (by reducing them mod `p`).

`int Flx_equal1(GEN x)` returns 1 (true) if the `Flx` x is equal to 1, 0 (false) otherwise.

`int Flx_equal(GEN x, GEN y)` returns 1 (true) if the `Flx` x and y are equal, and 0 (false) otherwise.

`GEN Flx_copy(GEN x)` returns a copy of `x`.

`GEN Flx_add(GEN x, GEN y, ulong p)`

`GEN Flx_Fl_add(GEN y, ulong x, ulong p)`

`GEN Flx_neg(GEN x, ulong p)`

`GEN Flx_neg_inplace(GEN x, ulong p)`, same as `Flx_neg`, in place (`x` is destroyed).

`GEN Flx_sub(GEN x, GEN y, ulong p)`

`GEN Flx_mul(GEN x, GEN y, ulong p)`

`GEN Flx_Fl_mul(GEN y, ulong x, ulong p)`

`GEN Flx_double(GEN y, ulong p)` returns $2y$.

`GEN Flx_triple(GEN y, ulong p)` returns $3y$.

`GEN Flx_mulu(GEN y, ulong x, ulong p)` as `Flx_Fl_mul` but do not assume that $x < p$.

`GEN Flx_Fl_mul_to_monic(GEN y, ulong x, ulong p)` returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

`GEN Flx_sqr(GEN x, ulong p)`

`GEN Flx_divrem(GEN x, GEN y, ulong p, GEN *pr)`

`GEN Flx_div(GEN x, GEN y, ulong p)`

`GEN Flx_rem(GEN x, GEN y, ulong p)`

`GEN Flx_deriv(GEN z, ulong p)`

`GEN Flx_gcd(GEN a, GEN b, ulong p)` returns a (not necessarily monic) greatest common divisor of x and y.

`GEN Flx_halfgcd(GEN x, GEN y, GEN p)` returns a two-by-two `FlxM` M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

`GEN Flx_extgcd(GEN a, GEN b, ulong p, GEN *ptu, GEN *ptv)`

`GEN Flx_pow(GEN x, long n, ulong p)`

`GEN Flx_roots(GEN f, ulong p)` returns the vector of roots of f (without multiplicity, as a `t_VECSMALL`). Assumes that p is prime.

`ulong Flx_oneroot(GEN f, ulong p)` returns one root $0 \leq r < p$ of the `Flx f` in $\mathbf{Z}/p\mathbf{Z}$. Return p if no root exists. Assumes that `p` is prime.

`GEN Flx_roots_naive(GEN f, ulong p)` returns the vector of roots of f as a `t_VECSMALL` (multiple roots are not repeated), found by an exhaustive search. Efficient for very small p !

`GEN Flx_factor(GEN f, ulong p)`

`GEN Flx_mod_Xn1(GEN T, ulong n, ulong p)` return T modulo $(X^n + 1, p)$. Shallow function.

`GEN Flx_mod_Xnm1(GEN T, ulong n, ulong p)` return T modulo $(X^n - 1, p)$. Shallow function.

`GEN Flx_degfact(GEN f, ulong p)` as `FpX_degfact`.

`GEN Flx_factorff_irred(GEN P, GEN Q, ulong p)` as `FpX_factorff_irred`.

`GEN Flx_ffisom(GEN P,GEN Q,ulong l)` as `FpX_ffisom`.

7.2.12.3 Miscellaneous operations.

`GEN pol0_Flx(long sv)` returns a zero `Flx` in variable v.

`GEN zero_Flx(long sv)` alias for `pol0_Flx`

`GEN pol1_Flx(long sv)` returns the unit `Flx` in variable v.

`GEN polx_Flx(long sv)` returns the variable v as degree 1 `Flx`.

`GEN Flx_normalize(GEN z, ulong p)`, as `FpX_normalize`.

`GEN random_Flx(long d, long sv, ulong p)` returns a random `Flx` in variable `v`, of degree less than `d`.

`GEN Flx_recip(GEN x)`, returns the reciprocal polynomial

`ulong Flx_resultant(GEN a, GEN b, ulong p)`, returns the resultant of `a` and `b`

`ulong Flx_extresultant(GEN a, GEN b, ulong p, GEN *ptU, GEN *ptV)` given two `Flx` `a` and `b`, returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set).

`GEN Flx_invBarrett(GEN T, ulong p)`, returns the Barrett inverse M of T defined by $M(x) \times x^n T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T.

`GEN Flx_renormalize(GEN x, long l)`, as `FpX_renormalize`, where `l` = `lg(x)`, in place.

`GEN Flx_shift(GEN T, long n)` returns `T` $* x^n$ if $n \geq 0$, and `T`$\backslash x^{-n}$ otherwise.

`long Flx_val(GEN x)` returns the valuation of `x`, i.e. the multiplicity of the 0 root.

`long Flx_valrem(GEN x, GEN *Z)` as `RgX_valrem`, returns the valuation of `x`. In particular, if the valuation is 0, set `*Z` to x, not a copy.

`GEN Flx_div_by_X_x(GEN A, ulong a, ulong p, ulong *rem)`, returns the Euclidean quotient of the `Flx` `A` by $X -$ `a`, and sets `rem` to the remainder `A`(`a`).

`ulong Flx_eval(GEN x, ulong y, ulong p)`, as `FpX_eval`.

`GEN Flx_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q.

`GEN Flx_splitting(GEN p, long k)`, as `RgX_splitting`.

`GEN Flx_inflate(GEN P, long d)` returns $P(X^d)$.

`int Flx_is_squarefree(GEN z, ulong p)`

`int Flx_is_irred(GEN f, ulong p)`, as `FpX_is_irred`.

`int Flx_is_smooth(GEN f, long r, ulong p)` return 1 if all irreducible factors of f are of degree at most r, 0 otherwise.

`long Flx_nbroots(GEN f, ulong p)`, as `FpX_nbroots`.

`long Flx_nbfact(GEN z, ulong p)`, as `FpX_nbfact`.

`GEN Flx_degfact(GEN f, ulong p)`, as `FpX_degfact`.

`GEN Flx_nbfact_by_degree(GEN z, long *nb, ulong p)` Assume that the `Flx` z is squarefree mod the prime p. Returns a `t_VECSMALL` D with $\deg z$ entries, such that $D[i]$ is the number of irreducible factors of degree i. Set `nb` to the total number of irreducible factors (the sum of the $D[i]$).

`void Flx_ffintersect(GEN P,GEN Q, long n, ulong p, GEN*SP, GEN*SQ, GEN MA,GEN MB)`, as `FpX_ffintersect`

`GEN Flv_polint(GEN x, GEN y, ulong p, long sv)` as `FpV_polint`, returning an `Flx` in variable v.

`GEN Flv_roots_to_pol(GEN a, ulong p, long sv)` as `FpV_roots_to_pol` returning an `Flx` in variable v.

7.2.13 `FlxV`. See `FpXV` operations.

`GEN FlxV_Flc_mul(GEN V, GEN W, ulong p)`, as `FpXV_FpC_mul`.

`GEN FlxV_red(GEN V, ulong p)` reduces each components with `Flx_red`.

7.2.14 `FlxT`. See `FpXT` operations.

`GEN FlxT_red(GEN V, ulong p)` reduces each leaf with `Flx_red`.

7.2.15 `Flxq`. See `FpXQ` operations.

`GEN Flxq_add(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sub(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sqr(GEN y, GEN T, ulong p)`

`GEN Flxq_inv(GEN x, GEN T, ulong p)`

`GEN Flxq_invsafe(GEN x, GEN T, ulong p)`

`GEN Flxq_div(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_pow(GEN x, GEN n, GEN T, ulong p)`

`GEN Flxq_powu(GEN x, ulong n, GEN T, ulong p)`

`GEN Flxq_powers(GEN x, long n, GEN T, ulong p)`

`GEN Flxq_matrix_pow(GEN x, long m, long n, GEN T, ulong p)`, see `FpXQ_matrix_pow`.

`GEN Flxq_autpow(GEN a, long n, GEN T, ulong p)` see `FpXQ_autpow`.

`GEN Flxq_autsum(GEN a, long n, GEN T, GEN p)` see `Flxq_autsum`.

`GEN Flxq_ffisom_inv(GEN S, GEN T, ulong p)`, as `FpXQ_ffisom_inv`.

`GEN Flx_Flxq_eval(GEN f, GEN x, GEN T, ulong p)` returns `f(x)`.

`GEN Flx_FlxqV_eval(GEN f, GEN x, GEN T, ulong p)`, see `FpX_FpXQV_eval`.

`GEN FlxqV_roots_to_pol(GEN V, GEN T, ulong p, long v)` as `FqV_roots_to_pol` returning an `FlxqX` in variable v.

`GEN Flxq_order(GEN a, GEN ord, GEN T, ulong p)` returns the order of the `t_Flxq a`. If `o` is non-`NULL`, it is assumed that `o` is a multiple of the order of `a`, either as a `t_INT` or a factorization matrix.

`int Flxq_issquare(GEN x, GEN T, ulong p)` returns 1 if x is a square and 0 otherwise. Assume that `T` is irreducible mod `p`.

`int Flxq_is2npower(GEN x, long n, GEN T, ulong p)` returns 1 if x is a 2^n-th power and 0 otherwise. Assume that `T` is irreducible mod `p`.

`GEN Flxq_log(GEN a, GEN g, GEN ord, GEN T, ulong p)` Let g of exact order `ord` in the field $F_p[X]/(T)$. Return e such that $a^e = g$. If e does not exists, the result is currently undefined. Assumes that `T` is irreducible mod `p`.

`GEN Flxq_sqrtn(GEN x, GEN n, GEN T, ulong p, GEN *zn)` returns an `n`-th root of `x`. Return `NULL` if `x` is not an `n`-th power residue. Otherwise, if `zn` is non-`NULL` set it to a primitive `n`-th root of 1. Assumes that `T` is irreducible mod `p`.

`GEN Flxq_sqrt(GEN x, GEN T, ulong p)` returns a square root of `x`. Return `NULL` if `x` is not a square.

`GEN Flxq_lroot(GEN a, GEN T, ulong p)` returns x such that $x^p = a$.

`GEN Flxq_lroot_fast(GEN a, GEN V, GEN T, ulong p)` assuming that `V=Flxq_powers(s,p-1,T,p)` where $s(x)^p \equiv x \pmod{T(x), p}$, returns b such that $b^p = a$. Only useful if p is less than the degree of T.

`GEN Flxq_charpoly(GEN x, GEN T, ulong p)` returns the characteristic polynomial of `x`

`GEN Flxq_minpoly(GEN x, GEN T, ulong p)` returns the minimal polynomial of `x`

`ulong Flxq_norm(GEN x, GEN T, ulong p)` returns the norm of `x`

`ulong Flxq_trace(GEN x, GEN T, ulong p)` returns the trace of `x`

`GEN Flxq_conjvec(GEN x, GEN T, ulong p)` returns the conjugates $[x, x^p, x^{p^2}, \ldots, x^{p^{n-1}}]$ where n is the degree of T.

`GEN gener_Flxq(GEN T, ulong p, GEN *po)` returns a primitive root modulo (T, p). T is an `Flx` assumed to be irreducible modulo the prime p. If `po` is not `NULL` it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

7.2.16 `FlxX`. See `FpXX` operations.

`GEN pol1_FlxX(long vX, long sx)` returns the unit `FlxX` as a `t_POL` in variable `vX` which only coefficient is `pol1_Flx(sx)`.

`GEN polx_FlxX(long vX, long sx)` returns the variable X as a degree 1 `t_POL` with `Flx` coefficients in the variable x.

`GEN FlxX_add(GEN P, GEN Q, ulong p)`

`GEN FlxX_sub(GEN P, GEN Q, ulong p)`

`GEN FlxX_Fl_mul(GEN x, ulong y, ulong p)`

`GEN FlxX_double(GEN x, ulong p)`

`GEN FlxX_triple(GEN x, ulong p)`

`GEN FlxX_neg(GEN x, ulong p)`

`GEN FlxX_Flx_add(GEN y, GEN x, ulong p)`

`GEN FlxX_Flx_mul(GEN x, GEN y, ulong p)`

`GEN FlxY_Flx_div(GEN x, GEN y, ulong p)` divides the coefficients of x by y using `Flx_div`.

`GEN FlxY_evalx(GEN x, ulong y, ulong p)`

`GEN FlxY_Flxq_evalx(GEN x, GEN y, ulong p)`

`GEN FlxX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`GEN FlxX_resultant(GEN u, GEN v, ulong p, long sv)` Returns $\mathrm{Res}_X(u, v)$, which is an `Flx`. The coefficients of `u` and `v` are assumed to be in the variable v.

`GEN Flx_FlxY_resultant(GEN a, GEN b, ulong p)` Returns $\mathrm{Res}_x(a, b)$, which is an `Flx` in the main variable of `b`.

`GEN FlxX_shift(GEN a, long n)`

`GEN FlxX_swap(GEN x, long n, long ws)`, as `RgXY_swap`.

`GEN FlxYqq_pow(GEN x, GEN n, GEN S, GEN T, ulong p)`, as `FpXYQQ_pow`.

7.2.17 `FlxqX`. See `FpXQX` operations.

`GEN zxX_to_Kronecker(GEN P, GEN Q)` assuming $P(X, Y)$ is a polynomial of degree in X strictly less than n, returns $P(X, X^{2*n-1})$, the Kronecker form of P.

`GEN Kronecker_to_FlxqX(GEN z, GEN T, ulong p)`. Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t)$ mod $(p, T(X))$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN FlxqX_red(GEN z, GEN T, ulong p)`

`GEN FlxqX_normalize(GEN z, GEN T, ulong p)`

`GEN FlxqX_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN FlxqX_Flxq_mul(GEN P, GEN U, GEN T, ulong p)`

`GEN FlxqX_Flxq_mul_to_monic(GEN P, GEN U, GEN T, ulong p)` returns $P * U$ assuming the result is monic of the same degree as P (in particular $U \neq 0$).

`GEN FlxqX_sqr(GEN x, GEN T, ulong p)`

`GEN FlxqX_pow(GEN x, long n, GEN T, ulong p)`

`GEN FlxqX_divrem(GEN x, GEN y, GEN T, ulong p, GEN *pr)`

`GEN FlxqX_div(GEN x, GEN y, GEN T, ulong p)`

`GEN FlxqX_rem(GEN x, GEN y, GEN T, ulong p)`

`GEN FlxqX_invBarrett(GEN T, GEN Q, ulong p)`

`GEN FlxqX_rem_Barrett(GEN x, GEN mg, GEN T, GEN Q, ulong p)`

`GEN FlxqX_gcd(GEN x, GEN y, ulong p)` returns a (not necessarily monic) greatest common divisor of x and y.

`GEN FlxqX_extgcd(GEN x, GEN y, GEN T, ulong p, GEN *ptu, GEN *ptv)`

`GEN FlxqXV_prod(GEN V, GEN T, ulong p)`

`GEN FlxqX_safegcd(GEN P, GEN Q, GEN T, ulong p)` Returns the *monic* GCD of P and Q if Euclid's algorithm succeeds and `NULL` otherwise. In particular, if p is not prime or T is not irreducible over $\mathbf{F}_p[X]$, the routine may still be used (but will fail if non-invertible leading terms occur).

`GEN FlxqX_Frobenius(GEN S, GEN T, GEN p)`, as `FpXQX_Frobenius`

`GEN FlxqXQ_halfFrobenius(GEN A, GEN S, GEN T, GEN p)`, as `FpXQXQ_halfFrobenius`

`long FlxqX_nbroots(GEN S, GEN T, GEN p)`, as `FpX_nbroots`.

`GEN FlxqX_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p)` as `FpX_FpXQ_eval`.

`GEN FlxqX_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p)` as `FpX_FpXQV_eval`.

7.2.18 `FlxqXQ`. See `FpXQXQ` operations.

`GEN FlxqXQ_mul(GEN x, GEN y, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_sqr(GEN x, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_inv(GEN x, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_div(GEN x, GEN y, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_pow(GEN x, GEN n, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_powers(GEN x, long n, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_matrix_pow(GEN x, long n, long m, GEN S, GEN T, ulong p)`

`GEN FlxqXQV_autpow(GEN a, long n, GEN S, GEN T, ulong p)` as `FpXQXQV_autpow`

`GEN FlxqXQV_autsum(GEN a, long n, GEN S, GEN T, ulong p)` as `FpXQXQV_autsum`

7.2.19 `F2x`. An `F2x z` is a `t_VECSMALL` representing a polynomial over $\mathbf{F}_2[X]$. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn(`v`)` for some variable v and the coefficients are given by the bits of remaining words by increasing degree.

7.2.19.1 Basic operations.

`ulong F2x_coeff(GEN x, long i)` returns the coefficient $i \geq 0$ of x.

`void F2x_clear(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 0.

`void F2x_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 0$ of x.

`void F2x_set(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 1.

`GEN Flx_to_F2x(GEN x)`

`GEN Z_to_F2x(GEN x, long v)`

`GEN ZX_to_F2x(GEN x)`

`GEN F2v_to_F2x(GEN x, long sv)`

`GEN ZXX_to_F2xX(GEN x, long v)`

`GEN F2x_to_Flx(GEN x)`

`GEN F2x_to_ZX(GEN x)`

`GEN pol0_F2x(long sv)` returns a zero `F2x` in variable v.

`GEN zero_F2x(long sv)` alias for `pol0_F2x`.

`GEN pol1_F2x(long sv)` returns the `F2x` in variable v constant to 1.

`GEN polx_F2x(long sv)` returns the variable v as degree 1 `F2x`.

`GEN random_F2x(long d, long sv)` returns a random `F2x` in variable `v`, of degree less than `d`.

`long F2x_degree(GEN x)` returns the degree of the `F2x x`. The degree of 0 is defined as -1.

`int F2x_equal1(GEN x)`

`int F2x_equal(GEN x, GEN y)`

`GEN F2x_1_add(GEN y)` returns `y+1` where `y` is a `Flx`.

`GEN F2x_add(GEN x, GEN y)`

`GEN F2x_mul(GEN x, GEN y)`

`GEN F2x_sqr(GEN x)`

`GEN F2x_divrem(GEN x, GEN y, GEN *pr)`

`GEN F2x_rem(GEN x, GEN y)`

`GEN F2x_div(GEN x, GEN y)`

`GEN F2x_renormalize(GEN x, long lx)`

`GEN F2x_deriv(GEN x)`

`GEN F2x_deflate(GEN x, long d)`

`void F2x_shift(GEN x, long d)` as `RgX_shift`

`void F2x_even_odd(GEN p, GEN *pe, GEN *po)` as `RgX_even_odd`

`long F2x_valrem(GEN x, GEN *Z)`

`GEN F2x_extgcd(GEN a, GEN b, GEN *ptu, GEN *ptv)`

`GEN F2x_gcd(GEN a, GEN b)`

`GEN F2x_halfgcd(GEN a, GEN b)`

`int F2x_issquare(GEN x)` returns 1 if x is a square of a `F2x` and 0 otherwise.

`int F2x_is_irred(GEN f)`, as `FpX_is_irred`.

`GEN F2x_sqrt(GEN x)` returns the squareroot of x, assuming x is a square of a `F2x`.

`GEN F2x_factor(GEN f)`

7.2.20 `F2xq`. See `FpXQ` operations.

`GEN F2xq_mul(GEN x, GEN y, GEN pol)`

`GEN F2xq_sqr(GEN x,GEN pol)`

`GEN F2xq_div(GEN x,GEN y,GEN T)`

`GEN F2xq_inv(GEN x, GEN T)`

`GEN F2xq_invsafe(GEN x, GEN T)`

`GEN F2xq_pow(GEN x, GEN n, GEN pol)`

`GEN F2xq_powu(GEN x, ulong n, GEN pol)`

`ulong F2xq_trace(GEN x, GEN T)`

`GEN F2xq_conjvec(GEN x, GEN T)` returns the vector of conjugates $[x, x^2, x^{2^2}, \ldots, x^{2^{n-1}}]$ where n is the degree of T.

`GEN F2xq_log(GEN a, GEN g, GEN ord, GEN T)`

`GEN F2xq_order(GEN a, GEN ord, GEN T)`

`GEN F2xq_Artin_Schreier(GEN a, GEN T)` returns a solution of $x^2 + x = a$, assuming it exists.

`GEN F2xq_sqrt(GEN a, GEN T)`

`GEN F2xq_sqrt_fast(GEN a, GEN s, GEN T)` assuming that $s^2 \equiv x \pmod{T(x)}$, computes $b \equiv a(s) \pmod{T}$ so that $b^2 = a$.

`GEN F2xq_sqrtn(GEN a, GEN n, GEN T, GEN *zeta)`

`GEN gener_F2xq(GEN T, GEN *po)`

`GEN F2xq_powers(GEN x, long n, GEN T)`

`GEN F2xq_matrix_pow(GEN x, long m, long n, GEN T)`

`GEN F2x_F2xq_eval(GEN f, GEN x, GEN T)`

`GEN F2x_F2xqV_eval(GEN f, GEN x, GEN T)`, see `FpX_FpXQV_eval`.

`GEN F2xq_autpow(GEN a, long n, GEN T)` computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_2[X]/T(X)$.

7.2.21 `F2xqV`, `F2xqM`.. See `FqV`, `FqM` operations.

`GEN F2xqM_F2xqC_mul(GEN a, GEN b, GEN T)`

`GEN F2xqM_ker(GEN x, GEN T)`

`GEN F2xqM_det(GEN a, GEN T)`

`GEN F2xqM_image(GEN x, GEN T)`

`GEN F2xqM_inv(GEN a, GEN T)`

`GEN F2xqM_mul(GEN a, GEN b, GEN T)`

`long F2xqM_rank(GEN x, GEN T)`

`GEN matid_F2xqM(long n, GEN T)`

7.2.22 Functions returning objects with t_INTMOD coefficients.

Those functions are mostly needed for interface reasons: t_INTMODs should not be used in library mode since the modular kernel is more flexible and more efficient, but GP users do not have access to the modular kernel. We document them for completeness:

GEN Fp_to_mod(GEN z, GEN p), z a t_INT. Returns z * Mod(1,p), normalized. Hence the returned value is a t_INTMOD.

GEN FpX_to_mod(GEN z, GEN p), z a ZX. Returns z * Mod(1,p), normalized. Hence the returned value has t_INTMOD coefficients.

GEN FpC_to_mod(GEN z, GEN p), z a ZC. Returns Col(z) * Mod(1,p), a t_COL with t_INTMOD coefficients.

GEN FpV_to_mod(GEN z, GEN p), z a ZV. Returns Vec(z) * Mod(1,p), a t_VEC with t_INTMOD coefficients.

GEN FpVV_to_mod(GEN z, GEN p), z a ZVV. Returns Vec(z) * Mod(1,p), a t_VEC of t_VEC with t_INTMOD coefficients.

GEN FpM_to_mod(GEN z, GEN p), z a ZM. Returns z * Mod(1,p), with t_INTMOD coefficients.

GEN F2c_to_mod(GEN x)

GEN F2m_to_mod(GEN x)

GEN Flc_to_mod(GEN z)

GEN Flm_to_mod(GEN z)

GEN FpXQC_to_mod(GEN V, GEN T, GEN p) V being a vector of FpXQ, converts each entry to a t_POLMOD with t_INTMOD coefficients, and return a t_COL.

GEN QXQV_to_mod(GEN V, GEN T) V a vector of QXQ, which are lifted representatives of elements of $\mathbf{Q}[X]/(T)$ (number field elements in most applications) and T is in $\mathbf{Z}[X]$. Return a vector where all non-rational entries are converted to t_POLMOD modulo T; no reduction mod T is attempted: the representatives should be already reduced. Used to normalize the output of nfroots.

GEN QXQXV_to_mod(GEN V, GEN T) V a vector of polynomials whose coefficients are QXQ. Analogous to QXQV_to_mod. Used to normalize the output of nffactor.

GEN QXQX_to_mod_shallow(GEN z, GEN T) v a polynomial with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

The following functions are obsolete and should not be used: they receive a polynomial with arbitrary coefficients, apply RgX_to_FpX, a function from the modular kernel, then *_to_mod:

GEN rootmod(GEN f, GEN p), applies FpX_roots.

GEN rootmod2(GEN f, GEN p), applies ZX_to_flx then Flx_roots_naive.

GEN factmod(GEN f, GEN p) applies FpX_factor.

GEN simplefactmod(GEN f, GEN p) applies FpX_degfact.

7.2.23 Chinese remainder theorem over Z.

`GEN Z_chinese(GEN a, GEN b, GEN A, GEN B)` returns the integer in $[0, \mathrm{lcm}(A, B)[$ congruent to a mod A and b mod B, assuming it exists; in other words, that a and b are congruent mod $\gcd(A, B)$.

`GEN Z_chinese_all(GEN a, GEN b, GEN A, GEN B, GEN *pC)` as `Z_chinese`, setting `*pC` to the lcm of A and B.

`GEN Z_chinese_coprime(GEN a, GEN b, GEN A, GEN B, GEN C)`, as `Z_chinese`, assuming that $\gcd(A, B) = 1$ and that $C = \mathrm{lcm}(A, B) = AB$.

`void Z_chinese_pre(GEN A, GEN B, GEN *pC, GEN *pU, GEN *pd)` initializes chinese remainder computations modulo A and B. Sets `*pC` to $\mathrm{lcm}(A, B)$, `*pd` to $\gcd(A, B)$, `*pU` to an integer congruent to 0 mod (A/d) and 1 mod (B/d). It is allowed to set `pd = NULL`, in which case, d is still computed, but not saved.

`GEN Z_chinese_post(GEN a, GEN b, GEN C, GEN U, GEN d)` returns the solution to the chinese remainder problem x congruent to a mod A and b mod B, where C, U, d were set in `Z_chinese_pre`. If d is `NULL`, assume the problem has a solution. Otherwise, return `NULL` if it has no solution.

The following pair of functions is used in homomorphic imaging schemes, when reconstructing an integer from its images modulo pairwise coprime integers. The idea is as follows: we want to discover an integer H which satisfies $|H| < B$ for some known bound B; we are given pairs (H_p, p) with H congruent to H_p mod p and all p pairwise coprime.

Given `H` congruent to H_p modulo a number of p, whose product is q, and a new pair (`Hp`, `p`), `p` coprime to q, the following incremental functions use the chinese remainder theorem (CRT) to find a new `H`, congruent to the preceding one modulo q, but also to `Hp` modulo `p`. It is defined uniquely modulo qp, and we choose the centered representative. When P is larger than $2B$, we have `H` $= H$, but of course, the value of `H` may stabilize sooner. In many applications it is possible to directly check that such a partial result is correct.

`GEN Z_init_CRT(ulong Hp, ulong p)` given a `Fl Hp` in $[0, p-1]$, returns the centered representative `H` congruent to `Hp` modulo `p`.

`int Z_incremental_CRT(GEN *H, ulong Hp, GEN *q, ulong p)` given a `t_INT *H`, centered modulo `*q`, a new pair (`Hp`, `p`) with `p` coprime to `q`, this function updates `*H` so that it also becomes congruent to (`Hp`, `p`), and `*q` to the product`qp` $=$ `p` $\cdot *$`q`. It returns 1 if the new value is equal to the old one, and 0 otherwise.

`GEN chinese1_coprime_Z(GEN v)` an alternative divide-and-conquer implementation: v is a vector of `t_INTMOD` with pairwise coprime moduli. Return the `t_INTMOD` solving the corresponding chinese remainder problem. This is a streamlined version of

`GEN chinese1(GEN v)`, which solves a general chinese remainder problem (not necessarily over **Z**, moduli not assumed coprime).

As above, for H a `ZM`: we assume that H and all `Hp` have dimension > 0. The original `*H` is destroyed.

`GEN ZM_init_CRT(GEN Hp, ulong p)`

`int ZM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above for H a `ZX`: note that the degree may increase or decrease. The original `*H` is destroyed.

`GEN ZX_init_CRT(GEN Hp, ulong p, long v)`

`int ZX_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

7.2.24 Rational reconstruction.

`int Fp_ratlift(GEN x, GEN m, GEN amax, GEN bmax, GEN *a, GEN *b)`. Assuming that $0 \leq x < m$, `amax` ≥ 0, and `bmax` > 0 are `t_INT`s, and that 2`amaxbmax` $< m$, attempts to recognize x as a rational a/b, i.e. to find `t_INT`s a and b such that

- $a \equiv bx$ modulo m,
- $|a| \leq$ `amax`, $0 < b \leq$ `bmax`,
- $\gcd(m, b) = \gcd(a, b)$.

If unsuccessful, the routine returns 0 and leaves a, b unchanged; otherwise it returns 1 and sets a and b.

In almost all applications, we actually know that a solution exists, as well as a non-zero multiple B of b, and $m = p^\ell$ is a prime power, for a prime p chosen coprime to B hence to b. Under the single assumption $\gcd(m, b) = 1$, if a solution a, b exists satisfying the three conditions above, then it is unique.

`GEN FpM_ratlift(GEN M, GEN m, GEN amax, GEN bmax, GEN denom)` given an `FpM` modulo m with reduced or `Fp_center`-ed entries, reconstructs a matrix with rational coefficients by applying `Fp_ratlift` to all entries. Assume that all preconditions for `Fp_ratlift` are satisfied, as well $\gcd(m, b) = 1$ (so that the solution is unique if it exists). Return `NULL` if the reconstruction fails, and the rational matrix otherwise. If `denom` is not `NULL` check further that all denominators divide `denom`.

The functions is not stack clean if one coefficients of M is negative (centered residues), but still suitable for `gerepileupto`.

`GEN FpX_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom)` as `FpM_ratlift`, where P is an `FpX`.

`GEN FpC_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom)` as `FpM_ratlift`, where P is an `FpC`.

7.2.25 Hensel lifts.

`GEN Zp_sqrtlift(GEN b, GEN a, GEN p, long e)` let a, b, p be `t_INT`s, with $p > 1$ odd, such that $a^2 \equiv b \bmod p$. Returns a `t_INT` A such that $A^2 \equiv b \bmod p^e$. Special case of `Zp_sqrtnlift`.

`GEN Zp_sqrtnlift(GEN b, GEN n, GEN a, GEN p, long e)` let a, b, n, p be `t_INT`s, with $n, p > 1$, and p coprime to n, such that $a^n \equiv b \bmod p$. Returns a `t_INT` A such that $A^n \equiv b \bmod p^e$. Special case of `ZpX_liftroot`.

`GEN Zp_teichmuller(GEN x, GEN p, long e, GEN pe)` for p an odd prime, x a `t_INT` coprime to p, and $pe = p^e$, returns the $(p-1)$-th root of 1 congruent to x modulo p, modulo p^e. For convenience, $p = 2$ is also allowed and we return 1 (x is 1 mod 4) or $2^e - 1$ (x is 3 mod 4).

`GEN ZpXQ_invlift(GEN b, GEN a, GEN T, GEN p, long e)` let p be a prime `t_INT` and a, b be `FpXQ`s (modulo T) such that $ab \equiv 1 \bmod (p, T)$. Returns an `FpXQ` A such that $Ab \equiv 1 \bmod (p^e, T)$. Special case of `ZpXQ_liftroot`.

`GEN ZpXQ_inv(GEN b, GEN T, GEN p, long e)` let p be a prime `t_INT` and b be a `FpXQ` (modulo T, p^e). Returns an `FpXQ` A such that $Ab \equiv 1 \bmod (p^e, T)$.

`GEN ZpXQ_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e)` let n, p be `t_INT`s, with $n, p > 1$ and p coprime to n, and a, b be `FpXQ`s (modulo T) such that $a^n \equiv b \bmod (p, T)$. Returns an `Fq` A such that $A^n \equiv b \bmod (p^e, T)$. Special case of `ZpXQ_liftroot`.

`GEN rootpadicfast(GEN f, GEN p, long e)` f a `ZX` with leading term prime to p, and without multiple roots mod p. Return a vector of `t_INT`s which are the roots of f mod p^e. This is a very important special case of `rootpadic`.

`GEN ZpX_liftroot(GEN f, GEN a, GEN p, long e)` f a `ZX` with leading term prime to p, and a a root mod p such that $v_p(f'(a)) = 0$. Return a `t_INT` which is the root of f mod p^e congruent to a mod p.

`GEN ZX_Zp_root(GEN f, GEN a, GEN p, long e)` same as `ZpX_liftroot` without the assumption $v_p(f'(a)) = 0$. Return a `t_VEC` of `t_INT`s, which are the p-adic roots of f congruent to a mod p (given modulo p^e).

`GEN ZpX_liftroots(GEN f, GEN S, GEN p, long e)` f a `ZX` with leading term prime to p, and S a vector of simple roots mod p. Return a vector of `t_INT`s which are the root of f mod p^e congruent to the $S[i]$ mod p.

`GEN ZpXQX_liftroot(GEN f, GEN a, GEN T, GEN p, long e)` as `ZpX_liftroot`, but f is now a polynomial in $\mathbf{Z}[X, Y]$ and we find a root in the unramified extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_p[Y]/(T)$, assuming $v_p(f(a)) > 0$ and $v_p(f'(a)) = 0$.

`GEN ZpXQX_liftroot_vald(GEN f, GEN a, long v, GEN T, GEN p, long e)` returns the foots of f as `ZpXQX_liftroot`, where v is the valuation of the content of f' and it is required that $v_p(f(a)) > v$ and $v_p(f'(a)) = v$.

`GEN ZpX_liftfact(GEN A, GEN B, GEN T, GEN p, long e, GEN pe)` is the routine underlying `polhensellift`. Here, p is prime, $T(Y)$ defines a finite field $\mathbf{F}_q$, either $\mathbf{F}_q = \mathbf{F}_p$ (T is `NULL`) or a non-prime finite field (T an `FpX`). A is a polynomial in $\mathbf{Z}[X]$ (T `NULL`) or $\mathbf{Z}[X, Y]$, whose leading coefficient is non-zero in $\mathbf{F}_q$. B is a vector of monic `FpX` (T `NULL`) or `FqX`, pairwise coprime in $\mathbf{F}_q[X]$, whose product is congruent to $A/\mathrm{lc}(A)$ in $\mathbf{F}_q[X]$. Lifts the elements of B mod `pe` $= p^e$, such that the congruence now holds mod (T, p^e).

The following technical function returns an optimal sequence of p-adic accuracies, for a given target accuracy:

`ulong quadratic_prec_mask(long n)` we want to reach accuracy $n \geq 1$, starting from accuracy 1, using a quadratically convergent, self-correcting, algorithm; in other words, from inputs correct to accuracy l one iteration outputs a result correct to accuracy $2l$. For instance, to reach $n = 9$, we want to use accuracies $[1, 2, 3, 5, 9]$ instead of $[1, 2, 4, 8, 9]$. The idea is to essentially double the accuracy at each step, and not overshoot in the end.

Let $a_0 = 1$, $a_1 = 2, \ldots, a_k = n$, be the desired sequence of accuracies. To obtain it, we work backwards and set

$$a_k = n, \quad a_{i-1} = (a_i + 1) \setminus 2.$$

This is in essence what the function returns. But we do not want to store the a_i explicitly, even as a `t_VECSMALL`, since this would leave an object on the stack. Instead, we store a_i implicitly in a bitmask `MASK`: let $a_0 = 1$, if the i-th bit of the mask is set, set $a_{i+1} = 2a_i - 1$, and $2a_i$ otherwise;

in short the bits indicate the places where we do something special and do not quite double the accuracy (which would be the straightforward thing to do).

In fact, to avoid returning separately the mask and the sequence length $k+1$, the function returns `MASK` $+ 2^{k+1}$, so the highest bit of the mask indicates the length of the sequence, and the following ones give an algorithm to obtain the accuracies. This is much simpler than it sounds, here is what it looks like in practice:

```
ulong mask = quadratic_prec_mask(n);
long l = 1;
while (mask > 1) {            /* here, the result is known to accuracy l */
  l = 2*l; if (mask & 1) l--; /* new accuracy l for the iteration */
  mask >>= 1;                 /* pop low order bit */
  /* ... lift to the new accuracy ... */
}
/* we are done. At this point l = n */
```

We just pop the bits in `mask` starting from the low order bits, stop when `mask` is 1 (that last bit corresponds to the 2^{k+1} that we added to the mask proper). Note that there is nothing specific to Hensel lifts in that function: it would work equally well for an Archimedean Newton iteration.

Note that in practice, we rather use an infinite loop, and insert an

```
if (mask == 1) break;
```

in the middle of the loop: the loop body usually includes preparations for the next iterations (e.g. lifting Bezout coefficients in a quadratic Hensel lift), which are costly and useless in the *last* iteration.

7.2.26 Other p-adic functions.

`long ZpX_disc_val(GEN f, GEN p)` returns the valuation at p of the discriminant of f. Assume that f is a monic *separable* `ZX` and that p is a prime number. Proceeds by dynamically increasing the p-adic accuracy; infinite loop if the discriminant of f is 0.

`long ZpX_resultant_val(GEN f, GEN g, GEN p, long M)` returns the valuation at p of $\mathrm{Res}(f,g)$. Assume f, g are both `ZX`, and that p is a prime number coprime to the leading coefficient of f. Proceeds by dynamically increasing the p-adic accuracy. To avoid an infinite loop when the resultant is 0, we return M if the Sylvester matrix mod p^M still does not have maximal rank.

`GEN ZpX_gcd(GEN f,GEN g, GEN p, GEN pm)` f a monic `ZX`, g a `ZX`, `pm` $= p^m$ a prime power. There is a unique integer $r \geq 0$ and a monic $h \in \mathbf{Q}_p[X]$ such that

$$p^r h\mathbf{Z}_p[X] + p^m\mathbf{Z}_p[X] = f\mathbf{Z}_p[X] + g\mathbf{Z}_p[X] + p^m\mathbf{Z}_p[X].$$

Return the 0 polynomial if $r \geq m$ and a monic $h \in \mathbf{Z}[1/p][X]$ otherwise (whose valuation at p is $> -m$).

`GEN ZpX_reduced_resultant(GEN f, GEN g, GEN p, GEN pm)` f a monic `ZX`, g a `ZX`, `pm` $= p^m$ a prime power. The p-adic *reduced resultant* of f and g is 0 if f, g not coprime in $\mathbf{Z}_p[X]$, and otherwise the generator of the form p^d of

$$(f\mathbf{Z}_p[X] + g\mathbf{Z}_p[X]) \cap \mathbf{Z}_p.$$

Return the reduced resultant modulo p^m.

`GEN ZpX_reduced_resultant_fast(GEN f, GEN g, GEN p, long M)` f a monic `ZX`, g a `ZX`, p a prime. Returns the the p-adic reduced resultant of f and g modulo p^M. This function computes resultants for a sequence of increasing p-adic accuracies (up to M p-adic digits), returning as soon as it obtains a non-zero result. It is very inefficient when the resultant is 0, but otherwise usually more efficient than computations using a priori bounds.

`GEN ZpM_echelon(GEN M, long early_abort, GEN p, GEN pm)` given a `ZM` M, a prime p and `pm` $= p^m$, returns an echelon form E for M mod p^m. I.e. there exist a square integral matrix U with $\det U$ coprime to p such that $E = MU$ modulo p^m. I `early_abort` is non-zero, return NULL as soon as one pivot in the echelon form is divisible by p^m. The echelon form is an upper triangular HNF, we do not waste time to reduce it to Gauss-Jordan form.

`GEN zlm_echelon(GEN M, long early_abort, ulong p, ulong pm)` variant of `ZpM_echelon`, for a `Zlm` M.

`GEN ZpXQ_log(GEN a, GEN T, GEN p, long e)` T being a `ZpX` irreducible modulo p, return the logarithm of a in $\mathbf{Z}_p[X]/(T)$ to precision e, assuming that $a \equiv 1 \pmod{p\mathbf{Z}_p[X]}$ if p odd or $a \equiv 1 \pmod{4\mathbf{Z}_2[X]}$ if $p = 2$.

`GEN padic_to_Q(GEN x)` truncate the `t_PADIC` to a `t_INT` or `t_FRAC`.

`GEN padic_to_Q_shallow(GEN x)` shallow version of `padic_to_Q`

`GEN QpV_to_QV(GEN v)` apply `padic_to_Q_shallow`

`long padicprec(GEN x, GEN p)` returns the absolute p-adic precision of the object x, by definition the minimum precision of the components of x. For a non-zero `t_PADIC`, this returns `valp(x) + precp(x)`.

`long padicprec_relative(GEN x)` returns the relative p-adic precision of the `t_INT`, `t_FRAC`, or `t_PADIC` x (minimum precision of the components of x for `t_POL` or vector/matrices). For a `t_PADIC`, this returns `precp(x)` if $x \neq 0$, and 0 for $x = 0$.

7.2.27 Conversions involving single precision objects.

7.2.27.1 To single precision.

`ulong Rg_to_Fl(GEN z, ulong p)`, `z` which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT`, a `t_INTMOD` whose modulus is divisible by p, a `t_FRAC` whose denominator is coprime to p, or a `t_PADIC` with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized, as an `Fl`.

`ulong Rg_to_F2(GEN z)`, as `Rg_to_Fl` for $p = 2$.

`ulong padic_to_Fl(GEN x, ulong p)` special case of `Rg_to_Fl`, for a x a `t_PADIC`.

`GEN RgX_to_F2x(GEN x)`, `x` a `t_POL`, returns the `F2x` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN RgX_to_Flx(GEN x, ulong p)`, `x` a `t_POL`, returns the `Flx` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN Rg_to_F2xq(GEN z, GEN T)`, `z` a `GEN` which can be mapped to $\mathbf{F}_2[X]/(T)$: anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_F2x` can be applied to, a `t_POLMOD` whose modulus is divisible by T (once mapped to a `F2x`), a suitable `t_RFRAC`. Returns `z` as an `F2xq`, normalized.

`GEN Rg_to_Flxq(GEN z, GEN T, ulong p)`, `z` a `GEN` which can be mapped to $\mathbf{F}_p[X]/(T)$: anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose

modulus is divisible by T (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns `z` as an `Flxq`, normalized.

`GEN ZX_to_Flx(GEN x, ulong p)` reduce `ZX x` modulo `p` (yielding an `Flx`). Faster than `RgX_to_Flx`.

`GEN ZV_to_Flv(GEN x, ulong p)` reduce `ZV x` modulo `p` (yielding an `Flv`).

`GEN ZXV_to_FlxV(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the vector's coefficients.

`GEN ZXT_to_FlxT(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the tree leaves.

`GEN ZXX_to_FlxX(GEN B, ulong p, long v)`, as `ZX_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN ZXXV_to_FlxXV(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the vector's coefficients.

`GEN RgC_to_Flc(GEN x, ulong p)` reduce the `t_VEC`/`t_COL` x modulo p, yielding a `t_VECSMALL`.

`GEN RgM_to_Flm(GEN x, ulong p)` reduce the `t_MAT` x modulo p.

`GEN ZM_to_Flm(GEN x, ulong p)` reduce `ZM` x modulo p (yielding an `Flm`).

`GEN ZV_to_zv(GEN z)`, converts coefficients using `itos`

`GEN ZV_to_nv(GEN z)`, converts coefficients using `itou`

`GEN ZM_to_zm(GEN z)`, converts coefficients using `itos`

`GEN FqC_to_FlxC(GEN x, GEN T, GEN p)`, converts coefficients in `Fq` to coefficient in Flx, result being a column vector.

`GEN FqV_to_FlxV(GEN x, GEN T, GEN p)`, converts coefficients in `Fq` to coefficient in Flx, result being a line vector.

`GEN FqM_to_FlxM(GEN x, GEN T, GEN p)`, converts coefficients in `Fq` to coefficient in Flx.

7.2.27.2 From single precision.

`GEN Flx_to_ZX(GEN z)`, converts to `ZX` (`t_POL` of non-negative `t_INT`s in this case)

`GEN Flx_to_FlxX(GEN z)`, converts to `FlxX` (`t_POL` of constant `Flx` in this case).

`GEN Flx_to_ZX_inplace(GEN z)`, same as `Flx_to_ZX`, in place (`z` is destroyed).

`GEN FlxX_to_ZXX(GEN B)`, converts an `FlxX` to a polynomial with `ZX` or `t_INT` coefficients (repeated calls to `Flx_to_ZX`).

`GEN FlxC_to_ZXC(GEN x)`, converts a vector of `Flx` to a column vector of polynomials with `t_INT` coefficients (repeated calls to `Flx_to_ZX`).

`GEN FlxV_to_ZXV(GEN x)`, as above but return a `t_VEC`.

`void F2xV_to_FlxV_inplace(GEN v)` v is destroyed.

`void F2xV_to_ZXV_inplace(GEN v)` v is destroyed.

`void FlxV_to_ZXV_inplace(GEN v)` v is destroyed.

`GEN FlxM_to_ZXM(GEN z)`, converts a matrix of `Flx` to a matrix of polynomials with `t_INT` coefficients (repeated calls to `Flx_to_ZX`).

`GEN zx_to_ZX(GEN z)`, as `Flx_to_ZX`, without assuming coefficients are non-negative.

`GEN Flc_to_ZC(GEN z)`, converts to `ZC` (`t_COL` of non-negative `t_INT`s in this case)

`GEN Flv_to_ZV(GEN z)`, converts to `ZV` (`t_VEC` of non-negative `t_INT`s in this case)

`GEN Flm_to_ZM(GEN z)`, converts to `ZM` (`t_MAT` with non-negative `t_INT`s coefficients in this case)

`GEN zc_to_ZC(GEN z)` as `Flc_to_ZC`, without assuming coefficients are non-negative.

`GEN zv_to_ZV(GEN z)` as `Flv_to_ZV`, without assuming coefficients are non-negative.

`GEN zm_to_ZM(GEN z)` as `Flm_to_ZM`, without assuming coefficients are non-negative.

`GEN zv_to_Flv(GEN z, ulong p)`

`GEN zm_to_Flm(GEN z, ulong p)`

7.2.27.3 Mixed precision linear algebra. Assumes dimensions are compatible. Multiply a multiprecision object by a single-precision one.

`GEN RgM_zc_mul(GEN x, GEN y)`

`GEN RgM_zm_mul(GEN x, GEN y)`

`GEN RgV_zc_mul(GEN x, GEN y)`

`GEN RgV_zm_mul(GEN x, GEN y)`

`GEN ZM_zc_mul(GEN x, GEN y)`

`GEN ZM_zm_mul(GEN x, GEN y)`

`GEN ZC_z_mul(GEN x, long y)`

`GEN ZM_nm_mul(GEN x, GEN y)` the entries of y are `ulong`s.

`GEN nm_Z_mul(GEN y, GEN c)` the entries of y are `ulong`s.

7.2.27.4 Miscellaneous involving Fl.

`GEN Fl_to_Flx(ulong x, long evx)` converts a `unsigned long` to a scalar `Flx`. Assume that `evx = evalvarn(vx)` for some variable number `vx`.

`GEN Z_to_Flx(GEN x, ulong p, long v)` converts a `t_INT` to a scalar polynomial in variable v.

`GEN Flx_to_Flv(GEN x, long n)` converts from `Flx` to `Flv` with `n` components (assumed larger than the number of coefficients of `x`).

`GEN zx_to_zv(GEN x, long n)` as `Flx_to_Flv`.

`GEN Flv_to_Flx(GEN x, long sv)` converts from vector (coefficient array) to (normalized) polynomial in variable v.

`GEN zv_to_zx(GEN x, long n)` as `Flv_to_Flx`.

`GEN Flm_to_FlxV(GEN x, long sv)` converts the columns of `Flm x` to an array of `Flx` in the variable v (repeated calls to `Flv_to_Flx`).

`GEN zm_to_zxV(GEN x, long n)` as `Flm_to_FlxV`.

`GEN Flm_to_FlxX(GEN x, long sw, long sv)` same as `Flm_to_FlxV(x,sv)` but returns the result as a (normalized) polynomial in variable w.

`GEN FlxV_to_Flm(GEN v, long n)` reverse `Flm_to_FlxV`, to obtain an `Flm` with `n` rows (repeated calls to `Flx_to_Flv`).

`GEN FlxX_to_Flm(GEN v, long n)` reverse `Flm_to_FlxX`, to obtain an `Flm` with `n` rows (repeated calls to `Flx_to_Flv`).

`GEN FlxX_to_FlxC(GEN B, long n, long sv)` see `RgX_to_RgV`. The coefficients of `B` are assumed to be in the variable v.

`GEN FlxXV_to_FlxM(GEN V, long n, long sv)` see `RgXV_to_RgM`. The coefficients of `V[i]` are assumed to be in the variable v.

`GEN Fly_to_FlxY(GEN a, long sv)` convert coefficients of `a` to constant `Flx` in variable v.

7.2.27.5 Miscellaneous involving `F2x`.

`GEN F2x_to_F2v(GEN x, long n)` converts from `F2x` to `F2v` with `n` components (assumed larger than the number of coefficients of `x`).

`GEN F2xC_to_ZXC(GEN x)`, converts a vector of `F2x` to a column vector of polynomials with `t_INT` coefficients (repeated calls to `F2x_to_ZX`).

`GEN F2xV_to_F2m(GEN v, long n)` `F2x_to_F2v` to each polynomial to get an `F2m` with `n` rows.

7.3 Higher arithmetic over Z: primes, factorization.

7.3.1 Pure powers.

`long Z_issquare(GEN n)` returns 1 if the `t_INT` n is a square, and 0 otherwise. This is tested first modulo small prime powers, then `sqrtremi` is called.

`long Z_issquareall(GEN n, GEN *sqrtn)` as `Z_issquare`. If n is indeed a square, set `sqrtn` to its integer square root. Uses a fast congruence test mod $64 \times 63 \times 65 \times 11$ before computing an integer square root.

`long Z_ispow2(GEN x)` returns 1 if the `t_INT` x is a power of 2, and 0 otherwise.

`long uissquare(ulong n)` as `Z_issquare`, for an `ulong` operand `n`.

`long uissquareall(ulong n, ulong *sqrtn)` as `Z_issquareall`, for an `ulong` operand `n`.

`ulong usqrt(ulong a)` returns the floor of the square root of a.

`ulong usqrtn(ulong a, ulong n)` returns the floor of the n-th root of a.

`long Z_ispower(GEN x, ulong k)` returns 1 if the `t_INT` n is a k-th power, and 0 otherwise; assume that $k > 1$.

`long Z_ispowerall(GEN x, ulong k, GEN *pt)` as `Z_ispower`. If n is indeed a k-th power, set `*pt` to its integer k-th root.

`long Z_isanypower(GEN x, GEN *ptn)` returns the maximal $k \geq 2$ such that the `t_INT` $x = n^k$ is a perfect power, or 0 if no such k exist; in particular `ispower(1)`, `ispower(0)`, `ispower(-1)` all return 0. If the return value k is not 0 (so that $x = n^k$) and `ptn` is not `NULL`, set `*ptn` to n.

The following low-level functions are called by `Z_isanypower` but can be directly useful:

`int is_357_power(GEN x, GEN *ptn, ulong *pmask)` tests whether the integer $x > 0$ is a 3-rd, 5-th or 7-th power. The bits of `*mask` initially indicate which test is to be performed; bit 0: 3-rd, bit 1: 5-th, bit 2: 7-th (e.g. $*$`pmask` $= 7$ performs all tests). They are updated during the call: if the "i-th power" bit is set to 0 then x is not a k-th power. The function returns 0 (not a 3-rd, 5-th or 7-th power), 3 (3-rd power, not a 5-th or 7-th power), 5 (5-th power, not a 7-th power), or 7 (7-th power); if an i-th power bit is initially set to 0, we take it at face value and assume x is not an i-th power without performing any test. If the return value k is non-zero, set `*ptn` to n such that $x = n^k$.

`int is_pth_power(GEN x, GEN *ptn, forprime_t *T, ulong cutoff)` let $x > 0$ be an integer, `cutoff` > 0 and T be an iterator over primes ≥ 11, we look for the smallest prime p such that $x = n^p$ (advancing T as we go along). The 11 is due to the fact that `is_357_power` and `issquare` are faster than the generic version for $p < 11$.

Fail and return 0 when the existence of p would imply $2^{\texttt{cutoff}} > x^{1/p}$, meaning that a possible n is so small that it should have been found by trial division; for maximal speed, you should start by a round of trial division, but the cut-off may also be set to 1 for a rigorous result without any trial division.

Otherwise returns the smallest suitable prime power p^i and set `*ptn` to the p^i-th root of x (which is now not a p-th power). We may immediately recall the function with the same parameters after setting $x = *$`ptn`: it will start at the next prime.

7.3.2 Factorization.

`GEN Z_factor(GEN n)` factors the `t_INT` `n`. The "primes" in the factorization are actually strong pseudoprimes.

`GEN absi_factor(GEN n)` returns `Z_factor(absi(n))`.

`long Z_issmooth(GEN n, ulong lim)` returns 1 if all the prime factors of the `t_INT` n are less or equal to lim.

`GEN Z_issmooth_fact(GEN n, ulong lim)` returns `NULL` if a prime factor of the `t_INT` n is $> lim$, and returns the factorization of n otherwise, as a `t_MAT` with `t_VECSMALL` columns (word-size primes and exponents). Neither memory-clean nor suitable for `gerepileupto`.

`GEN Z_factor_until(GEN n, GEN lim)` as `Z_factor`, but stop the factorization process as soon as the unfactored part is smaller than `lim`. The resulting factorization matrix only contains the factors found. No other assumptions can be made on the remaining factors.

`GEN Z_factor_limit(GEN n, ulong lim)` trial divide n by all primes $p <$ `lim` in the precomputed list of prime numbers and return the corresponding factorization matrix. In this case, the last "prime" divisor in the first column of the factorization matrix may well be a proven composite.

If `lim` $= 0$, the effect is the same as setting `lim` $=$ `maxprime`$() + 1$: use all precomputed primes.

`GEN absi_factor_limit(GEN n, ulong all)`returns `Z_factor_limit(absi(n))`.

`GEN boundfact(GEN x, ulong lim)` as `Z_factor_limit`, applying to `t_INT` or `t_FRAC` inputs.

`GEN Z_smoothen(GEN n, GEN L, GEN *pP, GEN *pE)` given a `t_VECSMALL` L containing a list of small primes and a `t_INT` n, trial divide n by the elements of L and return the cofactor. Return `NULL` if the cofactor is ± 1. `*P` and `*E` contain the list of prime divisors found and their exponents, as `t_VECSMALL`s. Neither memory-clean, nor suitable for `gerepileupto`.

`GEN Z_factor_listP(GEN N, GEN L)` given a `t_INT` N, a vector or primes L containing all prime divisors of N (and possibly others). Return `factor(N)`. Neither memory-clean, nor suitable for `gerepileupto`.

`GEN factor_pn_1(GEN p, ulong n)` returns the factorization of $p^n - 1$, where p is prime and n is a positive integer.

`GEN factor_pn_1_limit(GEN p, ulong n, ulong B)` returns a partial factorization of $p^n - 1$, where p is prime and n is a positive integer. Don't actively search for prime divisors $p > B$, but we may find still find some due to Aurifeuillian factorizations. Any entry $> B^2$ in the output factorization matrix is *a priori* not a prime (but may well be).

`GEN factor_Aurifeuille_prime(GEN p, long n)` an Aurifeuillian factor of $\phi_n(p)$, assuming p prime and an Aurifeuillian factor exists ($p\zeta_n$ is a square in $\mathbf{Q}(\zeta_n)$).

`GEN factor_Aurifeuille(GEN a, long d)` an Aurifeuillian factor of $\phi_n(a)$, assuming a is a non-zero integer and $n > 2$. Returns 1 if no Aurifeuillian factor exists.

`GEN odd_prime_divisors(GEN a) t_VEC` of all prime divisors of the `t_INT` a.

`GEN factoru(ulong n)`, returns the factorization of n. The result is a 2-component vector $[P, E]$, where P and E are `t_VECSMALL` containing the prime divisors of n, and the $v_p(n)$.

`GEN factoru_pow(ulong n)`, returns the factorization of n. The result is a 3-component vector $[P, E, C]$, where P, E and C are `t_VECSMALL` containing the prime divisors of n, the $v_p(n)$ and the $p^{v_p(n)}$.

`ulong tridiv_bound(GEN n)` returns the trial division bound used by `Z_factor`(n).

7.3.3 Checks associated to arithmetic functions.

Arithmetic functions accept arguments of the following kind: a plain positive integer N (`t_INT`), the factorization fa of a positive integer (a `t_MAT` with two columns containing respectively primes and exponents), or a vector $[N, fa]$. A few functions accept non-zero integers (e.g. `omega`), and some others arbitrary integers (e.g. `factorint`, ...).

`int is_Z_factorpos(GEN f)` returns 1 if f looks like the factorization of a positive integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are positive integers. It does *not* check that entries in the first column ("primes") are prime, or even pairwise coprime, nor that they are stricly increasing.

`int is_Z_factornon0(GEN f)` returns 1 if f looks like the factorization of a non-zero integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof, analogous to `is_Z_factorpos`. (Entries in the first column need only be non-zero integers.)

`int is_Z_factor(GEN f)` returns 1 if f looks like the factorization of an integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are integers. Entries in the second column ("exponents") are all positive. Either it encodes the "factorization" 0^e, $e > 0$, or entries in the first column ("primes") are all non-zero.

`GEN clean_Z_factor(GEN f)` assuming f is the factorization of an integer n, return the factorization of $|n|$, i.e. remove -1 from the factorization. Shallow function.

In the following two routines, f is the name of an arithmetic function, and n a supplied argument. They all raise exceptions if n does not correspond to an integer or an integer factorization of the expected shape.

`GEN check_arith_pos(GEN n, const char *f)` check whether n is associated to the factorization of a positive integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_DOMAIN` ($n \leq 0$) or an `e_TYPE` exception (other failures).

`GEN check_arith_non0(GEN n, const char *f)` check whether n is associated to the factorization of a non-0 integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_TYPE` exception.

`GEN check_arith_all(GEN n, const char *f)` is associated to the factorization of an integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise.

7.3.4 Incremental integer factorization.

Routines associated to the dynamic factorization of an integer n, iterating over successive prime divisors. This is useful to implement high-level routines allowed to take shortcuts given enough partial information: e.g. `moebius`(n) can be trivially computed if we hit p such that $p^2 \mid n$. For efficiency, trial division by small primes should have already taken place. In any case, the functions below assume that no prime $< 2^{14}$ divides n.

`GEN ifac_start(GEN n, int moebius)` schedules a new factorization attempt for the integer n. If `moebius` is non-zero, the factorization will be aborted as soon as a repeated factor is detected (Moebius mode). The function assumes that $n > 1$ is a *composite* `t_INT` whose prime divisors satisfy $p > 2^{14}$ *and* that one can write to n in place.

This function stores data on the stack, no `gerepile` call should delete this data until the factorization is complete. Returns `partial`, a data structure recording the partial factorization state.

`int ifac_next(GEN *partial, GEN *p, long *e)` deletes a primary factor p^e from `partial` and sets `p` (prime) and `e` (exponent), and normally returns 1. Whatever remains in the `partial` structure is now coprime to p.

Returns 0 if all primary factors have been used already, so we are done with the factorization. In this case p is set to `NULL`. If we ran in Moebius mode and the factorization was in fact aborted, we have $e = 1$, otherwise $e = 0$.

`int ifac_read(GEN part, GEN *k, long *e)` peeks at the next integer to be factored in the list k^e, where k is not necessarily prime and can be a perfect power as well, but will be factored by the next call to `ifac_next`. You can remove this factorization from the schedule by calling:

`void ifac_skip(GEN part)` removes the next scheduled factorization.

`int ifac_isprime(GEN n)` given n whose prime divisors are $> 2^{14}$, returns the decision the factoring engine would take about the compositeness of n: 0 if n is a proven composite, and 1 if we believe it to be prime; more precisely, n is a proven prime if `factor_proven` is set, and only a BPSW-pseudoprime otherwise.

7.3.5 Integer core, squarefree factorization.

`long Z_issquarefree(GEN n)` returns 1 if the `t_INT` `n` is square-free, and 0 otherwise.

`long Z_isfundamental(GEN x)` returns 1 if the `t_INT` `x` is a fundamental discriminant, and 0 otherwise.

`GEN core(GEN n)` unique squarefree integer d dividing n such that n/d is a square. The core of 0 is defined to be 0.

`GEN core2(GEN n)` return $[d, f]$ with d squarefree and $n = df^2$.

`GEN corepartial(GEN n, long lim)` as `core`, using `boundfact(n,lim)` to partially factor `n`. The result is not necessarily squarefree, but $p^2 \mid n$ implies $p >$ `lim`.

`GEN core2partial(GEN n, long lim)` as `core2`, using `boundfact(n,lim)` to partially factor `n`. The resulting d is not necessarily squarefree, but $p^2 \mid n$ implies $p >$ `lim`.

7.3.6 Primes, primality and compositeness tests.

7.3.6.1 Chebyshev's π function, bounds.

`ulong uprimepi(ulong n)`, returns the number of primes $p \leq n$ (Chebyshev's π function).

`double primepi_upper_bound(double x)` return a quick upper bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_upper_bound(GEN x)` as `primepi_upper_bound`, returns a `t_REAL`.

`double primepi_lower_bound(double x)` return a quick lower bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_lower_bound(GEN x)` as `primepi_lower_bound`, returns a `t_REAL` or `gen_0`.

7.3.6.2 Primes, primes in intervals.

`ulong unextprime(ulong n)`, returns the smallest prime $\geq n$. Return 0 if it cannot be represented as an `ulong` (n bigger than $2^{64} - 59$ or $2^{32} - 5$ depending on the word size).

`ulong uprecprime(ulong n)`, returns the largest prime $\leq n$. Return 0 if $n \leq 1$.

`ulong uprime(long n)` returns the n-th prime, assuming it fits in an `ulong` (overflow error otherwise).

`GEN prime(long n)` same as `utoi(uprime(n))`.

`GEN primes_zv(long m)` returns the first m primes, in a `t_VECSMALL`.

`GEN primes(long m)` return the first m primes, as a `t_VEC` of `t_INT`s.

`GEN primes_interval(GEN a, GEN b)` return the primes in the interval $[a, b]$, as a `t_VEC` of `t_INT`s.

`GEN primes_interval_zv(ulong a, ulong b)` return the primes in the interval $[a, b]$, as a `t_VECSMALL` of `ulong`ss.

`GEN primes_upto_zv(ulong b)` return the primes in the interval $[2, b]$, as a `t_VECSMALL` of `ulong`ss.

7.3.6.3 Tests.

`int uisprime(ulong p)`, returns 1 if `p` is a prime number and 0 otherwise.

`int isprime(GEN n)`, returns 1 if the `t_INT` `n` is a (fully proven) prime number and 0 otherwise.

`long isprimeAPRCL(GEN n)`, returns 1 if the `t_INT` `n` is a prime number and 0 otherwise, using only the APRCL test — not even trial division or compositeness tests. The workhorse `isprime` should be faster on average, especially if non-primes are included!

`long BPSW_psp(GEN n)`, returns 1 if the `t_INT` `n` is a Baillie-Pomerance-Selfridge-Wagstaff pseudoprime, and 0 otherwise (proven composite).

`int BPSW_isprime(GEN x)` assuming x is a BPSW-pseudoprime, rigorously prove its primality. The function `isprime` is currently implemented as

```
BPSW_psp(x) && BPSW_isprime(x)
```

`long millerrabin(GEN n, long k)` performs k strong Rabin-Miller compositeness tests on the `t_INT` n, using k random bases. This function also caches square roots of -1 that are encountered during the successive tests and stops as soon as three distinct square roots have been produced; we have in principle factored n at this point, but unfortunately, there is currently no way for the factoring machinery to become aware of it. (It is highly implausible that hard to find factors would be exhibited in this way, though.) This should be slower than `BPSW_psp` for $k \geq 4$ and we would expect it to be less reliable.

7.3.7 Iterators over primes.

`int forprime_init(forprime_t *T, GEN a, GEN b)` initialize an iterator T over primes in $[a, b]$; over primes $\geq a$ if $b =$ `NULL`. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise.

`GEN forprime_next(forprime_t *T)` returns the next prime in the range, assuming that T was initialized by `forprime_init`.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)`

`ulong u_forprime_next(forprime_t *T)`

`void u_forprime_restrict(forprime_t *T, ulong c)` let T an iterator over primes initialized via `u_forprime_init(&T, a, b)`, possibly followed by a number of calls to `u_forprime_next`, and $a \leq c \leq b$. Restrict the range of primes considered to $[a, c]$.

`int u_forprime_arith_init(forprime_t *T, ulong a,ulong b, ulong c,ulong q)` initialize an iterator over primes in $[a, b]$, congruent to c modulo q. Assume $0 \leq c < q$ and $(c, q) = 1$. Subsequent calls to `u_forprime_next` will only return primes congruent to c modulo q.

7.4 Integral, rational and generic linear algebra.

7.4.1 `ZC / ZV, ZM`. A `ZV` (resp. a `ZM`, resp. a `ZX`) is a `t_VEC` or `t_COL` (resp. `t_MAT`, resp. `t_POL`) with `t_INT` coefficients.

7.4.1.1 `ZC / ZV`.

`void RgV_check_ZV(GEN x, const char *s)` Assuming `x` is a `t_VEC` or `t_COL` raise an error if it is not a `ZV` (s should point to the name of the caller).

`int RgV_is_ZV(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a `ZV`, and 0 otherwise.

`int RgV_is_QV(GEN P)` return 1 is the `RgV` P has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int ZV_equal0(GEN x)` returns 1 if all entries of the `ZV` x are zero, and 0 otherwise.

`int ZV_cmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order, comparing absolute values).

`int ZV_abscmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order).

`int ZV_equal(GEN x, GEN y)` returns 1 if the two `ZV` are equal and 0 otherwise. A `t_COL` and a `t_VEC` with the same entries are declared equal.

`GEN ZC_add(GEN x, GEN y)` adds `x` and `y`.

`GEN ZC_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN ZC_Z_add(GEN x, GEN y)` adds `y` to `x[1]`.

`GEN ZC_Z_sub(GEN x, GEN y)` subtracts `y` to `x[1]`.

`GEN ZC_copy(GEN x)` returns a (`t_COL`) copy of `x`.

`GEN ZC_neg(GEN x)` returns $-$`x` as a `t_COL`.

`void ZV_neg_inplace(GEN x)` negates the `ZV x` in place, by replacing each component by its opposite (the type of `x` remains the same, `t_COL` or `t_COL`). If you want to save even more memory by avoiding the implicit component copies, use `ZV_togglesign`.

`void ZV_togglesign(GEN x)` negates `x` in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZC_Z_mul(GEN x, GEN y)` multiplies the `ZC` or `ZV x` (which can be a column or row vector) by the `t_INT y`, returning a `ZC`.

`GEN ZC_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZV_dotproduct(GEN x,GEN y)` as `RgV_dotproduct` assuming x and y have `t_INT` entries.

`GEN ZV_dotsquare(GEN x)` as `RgV_dotsquare` assuming x has `t_INT` entries.

`GEN ZC_lincomb(GEN u, GEN v, GEN x, GEN y)` returns $ux + vy$, where u, v are `t_INT` and x, y are `ZC` or `ZV`. Return a `ZC`

`void ZC_lincomb1_inplace(GEN X, GEN Y, GEN v)` sets $X \leftarrow X + vY$, where v is a `t_INT` and X, Y are `ZC` or `ZV`. (The result has the type of X.) Memory efficient (e.g. no-op if $v = 0$), but not gerepile-safe.

`GEN ZC_ZV_mul(GEN x, GEN y, GEN p)` multiplies the `ZC x` (seen as a column vector) by the `ZV y` (seen as a row vector, assumed to have compatible dimensions).

`GEN ZV_content(GEN x)` returns the GCD of all the components of `x`.

`GEN ZV_gcdext(GEN A)` given a vector of n integers A, returns $[d, U]$, where d is the content of A and U is a matrix in $\mathrm{GL}_n(\mathbf{Z})$ such that $AU = [D, 0, \ldots, 0]$.

`GEN ZV_prod(GEN x)` returns the product of all the components of `x` (1 for the empty vector).

`GEN ZV_sum(GEN x)` returns the sum of all the components of `x` (0 for the empty vector).

`long ZV_max_lg(GEN x)` returns the effective length of the longest entry in x.

`int ZV_dvd(GEN x, GEN y)` assuming x, y are two `ZV`s of the same length, return 1 if $y[i]$ divides $x[i]$ for all i and 0 otherwise. Error if one of the $y[i]$ is 0.

`GEN ZV_sort(GEN L)` sort the `ZV` L. Returns a vector with the same type as L.

`GEN ZV_sort_uniq(GEN L)` sort the `ZV` L, removing duplicate entries. Returns a vector with the same type as L.

`long ZV_search(GEN L, GEN y)` look for the `t_INT` y in the sorted `ZV` L. Return an index i such that $L[i] = y$, and 0 otherwise.

`GEN ZV_indexsort(GEN L)` returns the permutation which, applied to the `ZV` L, would sort the vector. The result is a `t_VECSMALL`.

`GEN ZV_union_shallow(GEN x, GEN y)` given two *sorted* ZV (as per `ZV_sort`, returns the union of x and y. Shallow function. In case two entries are equal in x and y, include the one from x.

7.4.1.2 `ZM`.

`void RgM_check_ZM(GEN A, const char *s)` Assuming `x` is a `t_MAT` raise an error if it is not a `ZM` (s should point to the name of the caller).

`GEN ZM_copy(GEN x)` returns a copy of `x`.

`int ZM_equal(GEN A, GEN B)` returns 1 if the two `ZM` are equal and 0 otherwise.

`GEN ZM_add(GEN x, GEN y)` returns `x + y` (assumed to have compatible dimensions).

`GEN ZM_sub(GEN x, GEN y)` returns `x − y` (assumed to have compatible dimensions).

`GEN ZM_neg(GEN x)` returns `−x`.

`void ZM_togglesign(GEN x)` negates `x` in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZM_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN ZM_Z_mul(GEN x, GEN y)` multiplies the `ZM x` by the `t_INT y`.

`GEN ZM_ZC_mul(GEN x, GEN y)` multiplies the `ZM x` by the `ZC y` (seen as a column vector, assumed to have compatible dimensions).

`GEN ZM_multosym(GEN x, GEN y)`

`GEN ZM_transmultosym(GEN x, GEN y)`

`GEN ZMrow_ZC_mul(GEN x, GEN y, long i)` multiplies the i-th row of `ZM x` by the `ZC y` (seen as a column vector, assumed to have compatible dimensions). Assumes that x is non-empty and $0 < i < \texttt{lg(x[1])}$.

`GEN ZV_ZM_mul(GEN x, GEN y)` multiplies the `ZV x` by the `ZM y`. Returns a `t_VEC`.

`GEN ZM_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZM_pow(GEN x, GEN n)` returns $\texttt{x}^{\texttt{n}}$, assuming `x` is a square `ZM` and $\texttt{n} \geq 0$.

`GEN ZM_powu(GEN x, ulong n)` returns $\texttt{x}^{\texttt{n}}$, assuming `x` is a square `ZM` and $\texttt{n} \geq 0$.

`GEN ZM_det(GEN M)` if `M` is a `ZM`, returns the determinant of M. This is the function underlying `matdet` whenever M is a `ZM`.

`GEN ZM_detmult(GEN M)` if `M` is a `ZM`, returns a multiple of the determinant of the lattice generated by its columns. This is the function underlying `detint`.

`GEN ZM_supnorm(GEN x)` return the sup norm of the `ZM` x.

`GEN ZM_charpoly(GEN M)` returns the characteristic polynomial (in variable 0) of the `ZM` M.

`GEN ZM_imagecompl(GEN x)` returns `matimagecompl(x)`.

`long ZM_rank(GEN x)` returns `matrank(x)`.

`GEN ZM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN ZM_indeximage(GEN x)` returns `gel(ZM_indexrank(x), 2)`.

`long ZM_max_lg(GEN x)` returns the effective length of the longest entry in x.

`GEN ZM_inv(GEN M, GEN d)` if `M` is a `ZM` and `d` is a `t_INT` such that $M' := \texttt{dM}^{-1}$ is integral, return M'. It is allowed to set `d = NULL`, in which case, the determinant of `M` is used instead.

`GEN QM_inv(GEN M, GEN d)` as above, with `M` a `QM`. We still assume that M' has integer coefficients.

`GEN ZM_det_triangular(GEN x)` returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`int ZM_isidentity(GEN x)` return 1 is the `ZM` x is the identity matrix, and 0 otherwise.

`int ZM_ishnf(GEN x)` return 1 if x is in HNF form, i.e. is upper triangular with positive diagonal coefficients, and for $j > i$, $x_{i,i} > x_{i,j} \geq 0$.

7.4.2 zv, zm.

`GEN zv_neg(GEN x)` return $-x$. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

`GEN zv_neg_inplace(GEN x)` negates x in place and return it. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

`GEN zm_zc_mul(GEN x, GEN y)`

`GEN zm_mul(GEN x, GEN y)`

`GEN zv_z_mul(GEN x, long n)` return nx. No check for overflow is done.

`long zv_content(GEN x)` returns the gcd of the entries of x.

`long zv_dotproduct(GEN x, GEN y)`

`long zv_prod(GEN x)` returns the product of all the components of `x` (assumes no overflow occurs).

`GEN zv_prod_Z(GEN x)` returns the product of all the components of `x`; consider all $x[i]$ as `ulong`s.

`long zv_sum(GEN x)` returns the sum of all the components of `x` (assumes no overflow occurs).

`int zv_cmp0(GEN x)` returns 1 if all entries of the `zv` x are 0, and 0 otherwise.

`int zv_equal(GEN x, GEN y)` returns 1 if the two `zv` are equal and 0 otherwise.

`int zv_equal0(GEN x)` returns 1 if all entries are 0, and return 0 otherwise.

`long zv_search(GEN L, long y)` look for y in the sorted `zv` L. Return an index i such that $L[i] = y$, and 0 otherwise.

`GEN zv_copy(GEN x)` as `Flv_copy`.

`GEN zm_transpose(GEN x)` as `Flm_transpose`.

`GEN zm_copy(GEN x)` as `Flm_copy`.

`GEN zero_zm(long m, long n)` as `zero_Flm`.

`GEN zero_zv(long n)` as `zero_Flv`.

`GEN row_zm(GEN A, long x0)` as `row_Flm`.

`int zvV_equal(GEN x, GEN y)` returns 1 if the two `zvV` (vectors of `zv`) are equal and 0 otherwise.

7.4.3 `ZMV / zmV` (vectors of `ZM/zm`).

`int RgV_is_ZMV(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if its components are `ZM`, and 0 otherwise.

`GEN ZMV_to_zmV(GEN z)`

`GEN zmV_to_ZMV(GEN z)`

7.4.4 `RgC / RgV, RgM`.

`RgC` and `RgV` routines assume the inputs are `VEC` or `COL` of the same dimension. `RgM` assume the inputs are `MAT` of compatible dimensions.

7.4.4.1 Matrix arithmetic.

`void RgM_dimensions(GEN)`x, long *m, long *n sets m, resp. n, to the number of rows, resp. columns of the `t_MAT` x.

`GEN RgC_add(GEN x, GEN y)` returns $x + y$ as a `t_COL`.

`GEN RgC_neg(GEN x)` returns $-x$ as a `t_COL`.

`GEN RgC_sub(GEN x, GEN y)` returns $x - y$ as a `t_COL`.

`GEN RgV_add(GEN x, GEN y)` returns $x + y$ as a `t_VEC`.

`GEN RgV_neg(GEN x)` returns $-x$ as a `t_VEC`.

`GEN RgV_sub(GEN x, GEN y)` returns $x - y$ as a `t_VEC`.

`GEN RgM_add(GEN x, GEN y)` return $x + y$.

`GEN RgM_neg(GEN x)` returns $-x$.

GEN RgM_sub(GEN x, GEN y) returns $x - y$.

GEN RgM_Rg_add(GEN x, GEN y) assuming x is a square matrix and y a scalar, returns the square matrix $x + y * \mathrm{Id}$.

GEN RgM_Rg_add_shallow(GEN x, GEN y) as RgM_Rg_add with much fewer copies. Not suitable for gerepileupto.

GEN RgM_Rg_sub(GEN x, GEN y) assuming x is a square matrix and y a scalar, returns the square matrix $x - y * \mathrm{Id}$.

GEN RgM_Rg_sub_shallow(GEN x, GEN y) as RgM_Rg_sub with much fewer copies. Not suitable for gerepileupto.

GEN RgC_Rg_add(GEN x, GEN y) assuming x is a non-empty column vector and y a scalar, returns the vector $[x_1 + y, x_2, \ldots, x_n]$.

GEN RgC_Rg_div(GEN x, GEN y)

GEN RgM_Rg_div(GEN x, GEN y) returns x/y (y treated as a scalar).

GEN RgC_Rg_mul(GEN x, GEN y)

GEN RgV_Rg_mul(GEN x, GEN y)

GEN RgM_Rg_mul(GEN x, GEN y) returns $x \times y$ (y treated as a scalar).

GEN RgV_RgC_mul(GEN x, GEN y) returns $x \times y$.

GEN RgV_RgM_mul(GEN x, GEN y) returns $x \times y$.

GEN RgM_RgC_mul(GEN x, GEN y) returns $x \times y$.

GEN RgM_mul(GEN x, GEN y) returns $x \times y$.

GEN RgM_transmul(GEN x, GEN y) returns $x^{\sim} \times y$.

GEN RgM_multosym(GEN x, GEN y) returns $x \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

GEN RgM_transmultosym(GEN x, GEN y) returns $x^{\sim} \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

GEN RgMrow_RgC_mul(GEN x, GEN y, long i) multiplies the i-th row of RgM x by the RgC y (seen as a column vector, assumed to have compatible dimensions). Assumes that x is non-empty and $0 < i < \mathtt{lg}(\mathtt{x}[1])$.

GEN RgM_mulreal(GEN x, GEN y) returns the real part of $x \times y$ (whose entries are t_INT, t_FRAC, t_REAL or t_COMPLEX).

GEN RgM_sqr(GEN x) returns x^2.

GEN RgC_RgV_mul(GEN x, GEN y) returns $x \times y$ (the square matrix $(x_i y_j)$).

The following two functions are not well defined in general and only provided for convenience in specific cases:

GEN RgC_RgM_mul(GEN x, GEN y) returns $x \times y[1,]$ if y is a row matrix $1 \times n$, error otherwise.

GEN RgM_RgV_mul(GEN x, GEN y) returns $x \times y[,1]$ if y is a column matrix $n \times 1$, error otherwise.

GEN RgM_powers(GEN x, long n) returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{n}}]$ as a t_VEC of RgMs.

`GEN RgV_sum(GEN v)` sum of the entries of v

`GEN RgV_sumpart(GEN v, long n)` returns the sum $v[1] + \ldots + v[n]$ (assumes that $\texttt{lg}(v) > n$).

`GEN RgV_sumpart2(GEN v, long m, long n)` returns the sum $v[m] + \ldots + v[n]$ (assumes that $\texttt{lg}(v) > n$ and $m > 0$). Returns `gen_0` when $m > n$.

`GEN RgV_dotproduct(GEN x,GEN y)` returns the scalar product of x and y

`GEN RgV_dotsquare(GEN x)` returns the scalar product of x with itself.

`GEN gram_matrix(GEN v)` returns the Gram matrix $(v_i \cdot v_j)$ associated to the entries of v (matrix, or vector of vectors).

`GEN RgV_polint(GEN X, GEN Y, long v)` X and Y being two vectors of the same length, returns the polynomial T in variable v such that $T(X[i]) = Y[i]$ for all i. The special case $X =$ `NULL` corresponds to $X = [1, 2, \ldots, n]$, where n is the length of Y.

7.4.4.2 Special shapes.

The following routines check whether matrices or vectors have a special shape, using `gequal1` and `gequal0` to test components. (This makes a difference when components are inexact.)

`int RgV_isscalar(GEN x)` return 1 if all the entries of x are 0 (as per `gequal0`), except possibly the first one. The name comes from vectors expressing polynomials on the standard basis $1, T, \ldots, T^{n-1}$, or on `nf.zk` (whose first element is 1).

`int QV_isscalar(GEN x)` as `RgV_isscalar`, assuming x is a `QV` (`t_INT` and `t_FRAC` entries only).

`int ZV_isscalar(GEN x)` as `RgV_isscalar`, assuming x is a `ZV` (`t_INT` entries only).

`int RgM_isscalar(GEN x, GEN s)` return 1 if x is the scalar matrix equal to s times the identity, and 0 otherwise. If s is `NULL`, test whether x is an arbitrary scalar matrix.

`int RgM_isidentity(GEN x)` return 1 is the `t_MAT` x is the identity matrix, and 0 otherwise.

`int RgM_isdiagonal(GEN x)` return 1 is the `t_MAT` x is a diagonal matrix, and 0 otherwise.

`int RgM_is_ZM(GEN x)` return 1 is the `t_MAT` x has only `t_INT` coefficients, and 0 otherwise.

`long RgV_isin(GEN v, GEN x)` return the first index i such that $v[i] = x$ if it exists, and 0 otherwise. Naive search in linear time, does not assume that `v` is sorted.

`GEN RgM_diagonal(GEN m)` returns the diagonal of m as a `t_VEC`.

`GEN RgM_diagonal_shallow(GEN m)` shallow version of `RgM_diagonal`

7.4.4.3 Conversion to floating point entries.

`GEN RgC_gtofp(GEN x, GEN prec)` returns the `t_COL` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgC_gtomp(GEN x, long prec)` returns the `t_COL` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x.

`GEN RgC_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgC_gtofp(x, prec) )
```

`GEN RgM_gtofp(GEN x, GEN prec)` returns the `t_MAT` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgM_gtomp(GEN x, long prec)` returns the `t_MAT` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x.

`GEN RgM_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgM_gtofp(x, prec) )
```

7.4.4.4 Linear algebra, linear systems.

`GEN RgM_inv(GEN a)` returns a left inverse of a (which needs not be square), or `NULL` if this turns out to be impossible. The latter happens when the matrix does not have maximal rank (or when rounding errors make it appear so).

`GEN RgM_inv_upper(GEN a)` as `RgM_inv`, assuming that a is a non-empty invertible upper triangular matrix, hence a little faster.

`GEN RgM_RgC_invimage(GEN A, GEN B)` returns a `t_COL` X such that $AX = B$ if one such exists, and `NULL` otherwise.

`GEN RgM_invimage(GEN A, GEN B)` returns a `t_MAT` X such that $AX = B$ if one such exists, and `NULL` otherwise.

`GEN RgM_Hadamard(GEN a)` returns a upper bound for the absolute value of $\det(a)$. The bound is a `t_INT`.

`GEN RgM_solve(GEN a, GEN b)` returns $a^{-1}b$ where a is a square `t_MAT` and b is a `t_COL` or `t_MAT`. Returns `NULL` if a^{-1} cannot be computed, see `RgM_inv`.

If $b =$ `NULL`, the matrix a need no longer be square, and we strive to return a left inverse for a (`NULL` if it does not exist).

`GEN RgM_solve_realimag(GEN M, GEN b)` M being a `t_MAT` with r_1+r_2 rows and r_1+2r_2 columns, y a `t_COL` or `t_MAT` such that the equation $Mx = y$ makes sense, returns x under the following simplifying assumptions: the first r_1 rows of M and y are real (the r_2 others are complex), and x is real. This is stabler and faster than calling `RgM_solve`(M, b) over $\mathbf{C}$. In most applications, M approximates the complex embeddings of an integer basis in a number field, and x is actually rational.

`GEN split_realimag(GEN x, long r1, long r2)` x is a `t_COL` or `t_MAT` with $r_1 + r_2$ rows, whose first r_1 rows have real entries (the r_2 others are complex). Return an object of the same type as x and $r_1 + 2r_2$ rows, such that the first $r_1 + r_2$ rows contain the real part of x, and the r_2 following ones contain the imaginary part of the last r_2 rows of x. Called by `RgM_solve_realimag`.

`GEN RgM_det_triangular(GEN x)` returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`GEN Frobeniusform(GEN V, long n)` given the vector V of elementary divisors for $M - x\mathrm{Id}$, where M is an $n \times n$ square matrix. Returns the Frobenius form of M. Used by `matfrobenius`.

`int RgM_QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec)` QR-decomposition of a square invertible `t_MAT` x with real coefficients. Sets `*pB` to the vector of squared lengths of the $x[i]$, `*pL` to the Gram-Schmidt coefficients and `*pQ` to a vector of successive Householder transforms. If R denotes the transpose of L and Q is the result of applying `*pQ` to the identity matrix, then $x = QR$ is the QR decomposition of x. Returns 0 is x is not invertible or we hit a precision problem, and 1 otherwise.

`int QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec)` as `RgM_QR_init`, assuming further that x has `t_INT` or `t_REAL` coefficients.

`GEN R_from_QR(GEN x, long prec)` assuming that x is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, return the upper triangular R from the QR docomposition of x. Not memory clean. If the matrix is not known to have `t_INT` or `t_REAL` coefficients, apply `RgM_gtomp` first.

`GEN gaussred_from_QR(GEN x, long prec)` assuming that x is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, returns `qfgaussred(x~* x)`; this is essentially the upper triangular R matrix from the QR decomposition of x, renormalized to accomodate `qfgaussred` conventions. Not memory clean.

7.4.5 Blackbox linear algebra.

A sparse column `zCs` v is a `t_COL` with two components C and E which are `t_VECSMALL` of the same length, representing $\sum_i E[i] * e_{C[i]}$, where (e_j) is the canonical basis. A sparse matrix (`zMs`) is a `t_VEC` of `zCs`.

`FpCs` and `FpMs` are identical to the above, but $E[i]$ is now interpreted as a *signed* C long integer representing an element of $\mathbf{F}_p$. This is important since p can be so large that $p + E[i]$ would not fit in a C long.

`RgCs` and `RgMs` are similar, except that the type of the components of E is now unspecified. Functions handling those later objects must not depend on the type of those components.

It is not possible to derive the space dimension (number of rows) from the above data. Thus most functions take an argument `nbrow` which is the number of rows of the corresponding column/matrix in dense representation.

`GEN zCs_to_ZC(GEN C, long nbrow)` convert the sparse vector C to a dense `ZC` of dimension `nbrow`.

`GEN zMs_to_ZM(GEN M, long nbrow)` convert the sparse matrix M to a dense `ZM` whose columns have dimension `nbrow`.

`GEN FpMs_FpC_mul(GEN M, GEN B, GEN p)` multiply the sparse matrix M (over $\mathbf{F}_p$) by the sparse vector B. The result is an `FpC`, i.e. a dense vector.

`GEN zMs_ZC_mul(GEN M, GEN B, GEN p)` multiply the sparse matrix M by the sparse vector B (over $\mathbf{Z}$). The result is an `ZC`, i.e. a dense vector.

`GEN FpV_FpMs_mul(GEN B, GEN M, GEN p)` multiply the sparse vector B by the sparse matrix M (over $\mathbf{F}_p$). The result is an `FpV`, i.e. a dense vector.

`GEN ZV_zMs_mul(GEN B, GEN M, GEN p)` multiply the sparse vector B (over $\mathbf{Z}$) by the sparse matrix M. The result is an `ZV`, i.e. a dense vector.

`void RgMs_structelim(GEN M, long nbrow, GEN A, GEN *p_col, GEN *p_row)` M being a RgMs with `nbrow` rows, A being a list of row indices, Perform structured elimination on M by removing some rows and columns until the number of effectively present rows is equal to the number of columns. the result is stored in two `t_VECSMALL`s, `*p_col` and `*p_row`: `*p_col` is a map from the new columns indices to the old one. `*p_row` is a map from the old rows indices to the new one (0 if removed).

`GEN FpMs_leftkernel_elt(GEN M, long nbrow, GEN p)` M being a sparse matrix over $\mathbf{F}_p$, return a non-zero kbdFpV X such that XM components are almost all 0.

`GEN FpMs_FpCs_solve(GEN M, GEN B, long nbrow, GEN p)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{F}_p$. Return either a solution as a `t_COL`

(dense vector), the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if B is not in the image of M).

`GEN FpMs_FpCs_solve_safe(GEN M, GEN B, long nbrow, GEN p)` as above, but in the event that p is not a prime and an impossible division occurs, return `NULL`.

`GEN ZpMs_ZpCs_solve(GEN M, GEN B, long nbrow, GEN p, long e)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{Z}/p^e\mathbf{Z}$. Return either a solution as a `t_COL` (dense vector), or the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component.

`GEN gen_FpM_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p)` solve the equation $f(X) = B$ over $\mathbf{F}_p$, where B is a `FpV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X. Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

`GEN gen_ZpM_Dixon(void *E, GEN (*f)(void*, GEN), GEN B, GEN p, long e)` solve equation $f(X) = B$ over $\mathbf{Z}/p^e\mathbf{Z}$, where B is a `ZV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X. Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

7.4.6 Obsolete functions.

The functions in this section are kept for backward compatibility only and will eventually disappear.

`GEN image2(GEN x)` compute the image of x using a very slow algorithm. Use `image` instead.

7.5 Integral, rational and generic polynomial arithmetic.

7.5.1 ZX.

`void RgX_check_ZX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it is not a `ZX` (s should point to the name of the caller).

`GEN ZX_copy(GEN x,GEN p)` returns a copy of `x`.

`long ZX_max_lg(GEN x)` returns the effective length of the longest component in x.

`GEN scalar_ZX(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x.

`GEN scalar_ZX_shallow(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x. Shallow function not suitable for `gerepile` and friends.

`GEN ZX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`int ZX_equal(GEN x, GEN y)` returns 1 if the two `ZX` have the same `degpol` and their coefficients are equal. Variable numbers are not checked.

`int ZX_equal1(GEN x)` returns 1 if the `ZX` is equal to 1 and 0 otherwise.

`GEN ZX_add(GEN x,GEN y)` adds `x` and `y`.

`GEN ZX_sub(GEN x,GEN y)` subtracts `x` and `y`.

`GEN ZX_neg(GEN x,GEN p)` returns $-$`x`.

`GEN ZX_Z_add(GEN x,GEN y)` adds the integer `y` to the `ZX` `x`.

`GEN ZX_Z_add_shallow(GEN x,GEN y)` shallow version of `ZX_Z_add`.

`GEN ZX_Z_sub(GEN x,GEN y)` subtracts the integer `y` to the `ZX` `x`.

`GEN Z_ZX_sub(GEN x,GEN y)` subtracts the `ZX` `y` to the integer `x`.

`GEN ZX_Z_mul(GEN x,GEN y)` multiplies the `ZX` `x` by the integer `y`.

`GEN ZX_mulu(GEN x, ulong y)` multiplies `x` by the integer `y`.

`GEN ZX_shifti(GEN x, long n)` shifts all coefficients of `x` by n bits, which can be negative.

`GEN ZX_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZX_remi2n(GEN x, long n)` reduces all coefficients of `x` to n bits, using `remi2n`.

`GEN ZX_mul(GEN x,GEN y)` multiplies `x` and `y`.

`GEN ZX_sqr(GEN x,GEN p)` returns $\mathtt{x}^2$.

`GEN ZX_mulspec(GEN a, GEN b, long na, long nb)`. Internal routine: `a` and `b` are arrays of coefficients representing polynomials $\sum_{i=0}^{\mathtt{na}-1} \mathtt{a}[i]X^i$ and $\sum_{i=0}^{\mathtt{nb}-1} \mathtt{b}[i]X^i$. Returns their product (as a true `GEN`).

`GEN ZX_sqrspec(GEN a, long na)`. Internal routine: `a` is an array of coefficients representing polynomial $\sum_{i=0}^{\mathtt{na}-1} \mathtt{a}[i]X^i$. Return its square (as a true `GEN`).

`GEN ZX_rem(GEN x, GEN y)` returns the remainder of the Euclidean division of x mod y. Assume that x, y are two `ZX` and that y is monic.

`GEN ZX_mod_Xnm1(GEN T, ulong n)` return T modulo $X^n - 1$). Shallow function.

`GEN ZX_gcd(GEN x,GEN y)` returns a gcd of the `ZX` x and y. Not memory-clean, but suitable for `gerepileupto`.

`GEN ZX_gcd_all(GEN x, GEN y, GEN *pX)`. returns a gcd d of x and y. If `pX` is not `NULL`, set $*$`pX` to a (non-zero) integer multiple of x/d. If x and y are both monic, then d is monic and `*pX` is exactly x/d. Not memory clean if the gcd is 1 (in that case `*pX` is set to x).

`GEN ZX_content(GEN x)` returns the content of the `ZX` x.

`long ZX_val(GEN P)` as `RgX_val`, but assumes `P` has `t_INT` coefficients.

`long ZX_valrem(GEN P, GEN *z)` as `RgX_valrem`, but assumes `P` has `t_INT` coefficients.

`GEN ZX_to_monic(GEN q GEN *L)` given q a non-zero `ZX`, returns a monic integral polynomial Q such that $Q(x) = Cq(x/L)$, for some rational C and positive integer $L > 0$. If `L` is not `NULL`, set `*L` to L; if $L = 1$, `*L` is set to `gen_1`. Not suitable for gerepileupto.

`GEN ZX_primitive_to_monic(GEN q, GEN *L)` as `ZX_to_monic` except q is assumed to have trivial content, which avoids recomputing it. The result is suboptimal if q is not primitive (L larger than necessary), but remains correct.

`GEN ZX_Z_normalize(GEN q, GEN *L)` a restricted version of `ZX_primitive_to_monic`, where q is a *monic* `ZX` of degree > 0. Finds the largest integer $L > 0$ such that $Q(X) := L^{-\deg q} q(Lx)$ is integral and return Q; this is not well-defined if q is a monomial, in that case, set $L = 1$ and $Q = q$. If `L` is not `NULL`, set `*L` to L.

`GEN ZX_Q_normalize(GEN q, GEN *L)` a variant of `ZX_Z_normalize` where $L > 0$ is allowed to be rational, the monic $Q \in \mathbf{Z}[X]$ has possibly smaller coefficients.

`GEN ZX_rescale(GEN P, GEN h)` returns $h^{\deg(P)}P(x/h)$. `P` is a `ZX` and `h` is a non-zero integer. Neither memory-clean nor suitable for `gerepileupto`.

`GEN ZX_rescale_lt(GEN P)` returns the monic integral polynomial $h^{\deg(P)-1}P(x/h)$, where `P` is a non-zero `ZX` and `h` is its leading coefficient. Neither memory-clean nor suitable for `gerepileupto`.

`GEN ZX_translate(GEN P, GEN c)` assume P is a `ZX` and c an integer. Returns $P(X+c)$ (optimized for $c = \pm 1$).

`GEN ZX_unscale(GEN P, GEN h)` given a `ZX` P and a `t_INT` h, returns $P(hx)$. Not memory clean.

`GEN ZX_unscale_div(GEN P, GEN h)` given a `ZX` P and a `t_INT` h such that $h \mid P(0)$, returns $P(hx)/h$. Not memory clean.

`GEN ZX_eval1(GEN P)` returns the integer $P(1)$.

`GEN ZX_graeffe(GEN p)` returns the Graeffe transform of p, i.e. the `ZX` q such that $p(x)p(-x) = q(x^2)$.

`GEN ZX_deriv(GEN x)` returns the derivative of `x`.

`GEN ZX_resultant(GEN A, GEN B)` returns the resultant of the `ZX` `A` and `B`.

`GEN ZX_disc(GEN T)` returns the discriminant of the `ZX` `T`.

`GEN ZX_factor(GEN T)` returns the factorization of the primitive part of `T` over $\mathbf{Q}[X]$ (the content is lost).

`int ZX_is_squarefree(GEN T)` returns 1 if the `ZX` `T` is squarefree, 0 otherwise.

`long ZX_is_irred(GEN T)` returns 1 it `T` is irreducible, and 0 otherwise.

`GEN ZX_squff(GEN T, GEN *E)` write T as a product $\prod T_i^{e_i}$ with the $e_1 < e_2 < \cdots$ all distinct and the T_i pairwise coprime. Return the vector of the T_i, and set `*E` to the vector of the e_i, as a `t_VECSMALL`.

7.5.2 ZXQ.

`GEN ZXQ_mul(GEN x,GEN y,GEN T)` returns $x * y \bmod T$, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQ_sqr(GEN x,GEN T)` returns $x^2 \bmod T$, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQ_charpoly(GEN A, GEN T, long v)`: let `T` and `A` be `ZX`s, returns the characteristic polynomial of `Mod(A, T)`. More generally, `A` is allowed to be a `QX`, hence possibly has rational coefficients, *assuming* the result is a `ZX`, i.e. the algebraic number `Mod(A,T)` is integral over `Z`.

`GEN ZX_ZXY_resultant(GEN A, GEN B)` under the assumption that `A` in $\mathbf{Z}[Y]$, `B` in $\mathbf{Q}[Y][X]$, and $R = \mathrm{Res}_Y(A, B) \in \mathbf{Z}[X]$, returns the resultant R.

`GEN ZX_compositum_disjoint(GEN A, GEN B)` given two irreducible `ZX` defining linearly disjoint extensions, returns a `ZX` defining their compositum.

`GEN ZX_ZXY_rnfequation(GEN A, GEN B, long *lambda)`, assume `A` in $\mathbf{Z}[Y]$, `B` in $\mathbf{Q}[Y][X]$, and $R = \mathrm{Res}_Y(A, B) \in \mathbf{Z}[X]$. If `lambda = NULL`, returns R as in `ZY_ZXY_resultant`. Otherwise, `lambda`

must point to some integer, e.g. 0 which is used as a seed. The function then finds a small $\lambda \in \mathbf{Z}$ (starting from `*lambda`) such that $R_\lambda(X) := \mathrm{Res}_Y(A, B(X + \lambda Y))$ is squarefree, resets `*lambda` to the chosen value and returns R_λ.

7.5.3 ZXV.

`GEN ZXV_equal(GEN x,GEN y)` returns 1 if the two vectors of `ZX` are equal, as per `ZX_equal` (variables are not checked to be equal) and 0 otherwise.

`GEN ZXV_Z_mul(GEN x,GEN y)` multiplies the vector of `ZX` `x` by the integer `y`.

`GEN ZXV_remi2n(GEN x, long n)` applies `ZX_remi2n` to all coefficients of `x`.

`GEN ZXV_dotproduct(GEN x,GEN y)` as `RgV_dotproduct` assuming x and y have `ZX` entries.

7.5.4 ZXT.

`GEN ZXT_remi2n(GEN x, long n)` applies `ZX_remi2n` to all leaves of the tree `x`.

7.5.5 ZXX.

`void RgX_check_ZXX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it one of its coefficients is not an integer or a `ZX` (s should point to the name of the caller).

`GEN ZXX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`long ZXX_max_lg(GEN x)` returns the effective length of the longest component in x; assume all coefficients are `t_INT` or `ZX`s.

`GEN ZXX_Z_divexact(GEN x, GEN y)` returns x/y assuming all integer divisions are exact.

`GEN ZXX_to_Kronecker(GEN P, long n)` Assuming $P(X, Y)$ is a polynomial of degree in X strictly less than n, returns $P(X, X^{2*n-1})$, the Kronecker form of P. Shallow function.

`GEN ZXX_to_Kronecker_spec(GEN Q, long lQ, long n)` return `ZXX_to_Kronecker`(P, n), where P is the polynomial $\sum_{i=0}^{\mathtt{lQ}-1} Q[i]x^i$. To be used when splitting the coefficients of genuine polynomials into blocks. Shallow function.

`GEN Kronecker_to_ZXX(GEN z, long n, long v)` recover $P(X, Y)$ from its Kronecker form $P(X, X^{2*n-1})$, v is the variable number corresponding to Y. Shallow function.

`GEN ZXX_mul_Kronecker(GEN P, GEN Q, long n)` return `ZX_mul` applied to the Kronecker forms $P(X, X^{2n-1})$ and $Q(X, X^{2n-1}$ of P and Q. Not memory clean.

7.5.6 QX.

`void RgX_check_QX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it is not a `QX` (s should point to the name of the caller).

`GEN QX_gcd(GEN x,GEN y)` returns a gcd of the `QX` x and y.

`GEN QX_disc(GEN T)` returns the discriminant of the `QX` `T`.

`GEN QX_factor(GEN T)` as `ZX_factor`.

`GEN QX_resultant(GEN A, GEN B)` returns the resultant of the `QX` `A` and `B`.

`GEN QX_complex_roots(GEN p, long l)` returns the complex roots of the `QX` p at accuracy l, where real roots are returned as `t_REAL`s. More efficient when p is irreducible and primitive. Special case of `cleanroots`.

7.5.7 QXQ.

`GEN QXQ_norm(GEN A, GEN B)` A being a `QX` and B being a `ZX`, returns the norm of the algebraic number $A \bmod B$, using a modular algorithm. To ensure that B is a `ZX`, one may replace it by `Q_primpart(B)`, which of course does not change the norm.

If A is not a `ZX` — it has a denominator —, but the result is nevertheless known to be an integer, it is much more efficient to call `QXQ_intnorm` instead.

`GEN QXQ_intnorm(GEN A, GEN B)` A being a `QX` and B being a `ZX`, returns the norm of the algebraic number $A \bmod B$, *assuming* that the result is an integer, which is for instance the case is $A \bmod B$ is an algebraic integer, in particular if A is a `ZX`. To ensure that B is a `ZX`, one may replace it by `Q_primpart(B)` (which of course does not change the norm).

If the result is not known to be an integer, you must use `QXQ_norm` instead, which is slower.

`GEN QXQ_inv(GEN A, GEN B)` returns the inverse of A modulo B where A is a `QX` and B is a `ZX`. Should you need this for a `QX` B, just use

```
QXQ_inv(A, Q_primpart(B));
```

But in all cases where modular arithmetic modulo B is desired, it is much more efficient to replace B by `Q_primpart`(B) once and for all.

`GEN QXQ_powers(GEN x, long n, GEN T)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^n]$ as `RgXQ_powers` would, but in a more efficient way when x has a huge integer denominator (we start by removing that denominator). Meant to be used to precompute powers of algebraic integers in $\mathbf{Q}[t]/(T)$. The current implementation does not require x to be a `QX`: any polynomial to which `Q_remove_denom` can be applied is fine.

`GEN QXQ_reverse(GEN f, GEN T)` as `RgXQ_reverse`, assuming f is a `QX`.

`GEN QX_ZXQV_eval(GEN f, GEN nV, GEN dV)` as `RgX_RgXQV_eval`, except that f is assumed to be a `QX`, V is given implicitly by a numerator `nV` (`ZV`) and denominator `dV` (a positive `t_INT` or `NULL` for trivial denominator). Not memory clean, but suitable for `gerepileupto`.

`GEN QXV_QXQ_eval(GEN v, GEN a, GEN T)` v is a vector of `QX`s (possibly scalars, i.e. rational numbers, for convenience), a and T both `QX`. Return the vector of evaluations at a modulo T. Not memory clean, nor suitable for `gerepileupto`.

`GEN QXX_QXQ_eval(GEN P, GEN a, GEN T)` $P(X, Y)$ is a `t_POL` with `QX` coefficients (possibly scalars, i.e. rational numbers, for convenience) , a and T both `QX`. Return the `QX` $P(X, a \bmod T)$. Not memory clean, nor suitable for `gerepileupto`.

`GEN nfgcd(GEN P, GEN Q, GEN T, GEN den)` given P and Q in $\mathbf{Z}[X, Y]$, T monic irreducible in $\mathbf{Z}[Y]$, returns the primitive d in $\mathbf{Z}[X, Y]$ which is a gcd of P, Q in $K[X]$, where K is the number field $\mathbf{Q}[Y]/(T)$. If not `NULL`, `den` is a multiple of the integral denominator of the (monic) gcd of P, Q in $K[X]$.

`GEN nfgcd_all(GEN P, GEN Q, GEN T, GEN den, GEN *Pnew)` as `nfgcd`. If `Pnew` is not `NULL`, set `*Pnew` to a non-zero integer multiple of P/d. If P and Q are both monic, then d is monic and `*Pnew` is exactly P/d. Not memory clean if the gcd is 1 (in that case `*Pnew` is set to P).

7.5.8 zx.

`GEN zero_zx(long sv)` returns a zero `zx` in variable v.

`GEN polx_zx(long sv)` returns the variable v as degree 1 `Flx`.

`GEN zx_renormalize(GEN x, long l)`, as `Flx_renormalize`, where `l` $=$ `lg(x)`, in place.

`GEN zx_shift(GEN T, long n)` returns `T` multiplied by `x`^{n}, assuming $n \geq 0$.

7.5.9 RgX.

`long RgX_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the polynomial x is defined. Raise an error if it detects consistency problems in modular objects: incompatible rings (e.g. $\mathbf{F}_p$ and $\mathbf{F}_q$ for primes $p \neq q$, $\mathbf{F}_p[X]/(T)$ and $\mathbf{F}_p[X]/(U)$ for $T \neq U$). Minor discrepancies are supported if they make general sense (e.g. $\mathbf{F}_p$ and $\mathbf{F}_{p^k}$, but not $\mathbf{F}_p$ and $\mathbf{Q}_p$); `t_FFELT` and `t_POLMOD` of `t_INTMOD`s are considered inconsistent, even if they define the same field: if you need to use simultaneously these different finite field implementations, multiply the polynomial by a `t_FFELT` equal to 1 first.

- 0: none of the others (presumably multivariate, possibly inconsistent).
- `t_INT`: defined over $\mathbf{Q}$ (not necessarily $\mathbf{Z}$).
- `t_INTMOD`: defined over $\mathbf{Z}/p\mathbf{Z}$, where `*ptp` is set to p. It is not checked whether p is prime.
- `t_COMPLEX`: defined over $\mathbf{C}$ (at least one `t_COMPLEX` with at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_REAL`: defined over $\mathbf{R}$ (at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_PADIC`: defined over $\mathbf{Q}_p$, where `*ptp` is set to p and `*ptprec` to the p-adic accuracy.
- `t_FFELT`: defined over a finite field $\mathbf{F}_{p^k}$, where `*ptp` is set to the field characteristic p and `*ptpol` is set to a `t_FFELT` belonging to the field.
- other values are composite corresponding to quotients $R[X]/(T)$, with one primary type `t1`, describing the form of the quotient, and a secondary type `t2`, describing R. If `t` is the `RgX_type`, `t1` and `t2` are recovered using

`void RgX_type_decode(long t, long *t1, long *t2)`

`t1` is one of

`t_POLMOD`: at least one `t_POLMOD` component, set `*ppol` to the modulus,

`t_QUAD`: no `t_POLMOD`, at least one `t_QUAD` component, set `*ppol` to the modulus (`-.pol`) of the `t_QUAD`,

`t_COMPLEX`: no `t_POLMOD` or `t_QUAD`, at least one `t_COMPLEX` component, set `*ppol` to $y^2 + 1$.

and the underlying base ring R is given by `t2`, which is one of `t_INT`, `t_INTMOD` (set `*ptp`) or `t_PADIC` (set `*ptp` and `*ptprec`), with the same meaning as above.

`int RgX_type_is_composite(long t)` t as returned by `RgX_type`, return 1 if t is a composite type, and 0 otherwise.

`GEN RgX_get_0(GEN x)` returns 0 in the base ring over which x is defined, to the proper accuracy (e.g. `0`, `Mod(0,3)`, `O(5^10)`).

`GEN RgX_get_1(GEN x)` returns 1 in the base ring over which x is defined, to the proper accuracy (e.g. `0`, `Mod(0,3)`,

`int RgX_isscalar(GEN x)` return 1 if x all the coefficients of x of degree > 0 are 0 (as per `gequal0`).

`int RgX_blocks(GEN P, long n, long m)` writes $P(X) = a_0(X) + X^n * a_1(X) * X^n + \ldots + X^{n*(m-1)}a_{m-1}(X)$, where the a_i are polynomial of degree at most $n-1$ (except possibly for the last one) and returns $[a_0(X), a_1(X), \ldots, a_{m-1}(X)]$. This is a shallow function.

`void RgX_even_odd(GEN p, GEN *pe, GEN *po)` write $p(X) = E(X^2) + XO(X^2)$ and set `*pe = E`, `*po = O`.

`GEN RgX_splitting(GEN P, long k)` write $P(X) = a_0(X^k) + Xa_1(X^k) + \ldots + X^{k-1}a_{k-1}(X^k)$ and return $[a_0(X), a_1(X), \ldots, a_{k-1}(X)]$. This is a shallow function.

`GEN RgX_copy(GEN x)` returns (a deep copy of) `x`.

`GEN RgX_add(GEN x,GEN y)` adds `x` and `y`.

`GEN RgX_sub(GEN x,GEN y)` subtracts `x` and `y`.

`GEN RgX_neg(GEN x)` returns $-$`x`.

`GEN RgX_Rg_add(GEN y, GEN x)` returns $x + y$.

`GEN RgX_Rg_add_shallow(GEN y, GEN x)` returns $x + y$; shallow function.

`GEN Rg_RgX_sub(GEN x, GEN y)`

`GEN RgX_Rg_sub(GEN y, GEN x)` returns $x - y$

`GEN RgX_mul(GEN x, GEN y)` multiplies the two `t_POL` (in the same variable) `x` and `y`. Uses Karatsuba algorithm.

`GEN RgX_mul_normalized(GEN A, long a, GEN B, long b)` returns $(X^a+A)(X^b+B)-X^(a+b)$, where we assume that $\deg A < a$ and $\deg B < b$ are polynomials in the same variable X.

`GEN RgX_mulspec(GEN a, GEN b, long na, long nb)`. Internal routine: `a` and `b` are arrays of coefficients representing polynomials $\sum_{i=0}^{\mathtt{na}-1} \mathtt{a}[i]X^i$ and $\sum_{i=0}^{\mathtt{nb}-1} \mathtt{b}[i]X^i$. Returns their product (as a true `GEN`).

`GEN RgX_mullow(GEN a, GEN b, long n)` returns ab modulo X^n, where a, b are two `t_POL` in the same variable X and $n \geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

`GEN RgX_sqr(GEN x)` squares the `t_POL` `x`. Uses Karatsuba algorithm.

`GEN RgX_sqrspec(GEN a, long na)`. Internal routine: `a` is an array of coefficients representing polynomial $\sum_{i=0}^{\mathtt{na}-1} \mathtt{a}[i]X^i$. Return its square (as a true `GEN`).

`GEN RgX_sqrlow(GEN a, long n)` returns a^2 modulo X^n, where a is a `t_POL` in the variable X and $n \geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

`GEN RgX_divrem(GEN x, GEN y, GEN *r)` by default, returns the Euclidean quotient and store the remainder in r. Three special values of r change that behavior • `NULL`: do not store the remainder, used to implement `RgX_div`,

- `ONLY_REM`: return the remainder, used to implement `RgX_rem`,

- ONLY_DIVIDES: return the quotient if the division is exact, and NULL otherwise.

GEN RgX_div(GEN x, GEN y)

GEN RgX_div_by_X_x(GEN A, GEN a, GEN *r) returns the quotient of the RgX A by $(X - a)$, and sets r to the remainder A(a).

GEN RgX_rem(GEN x, GEN y)

GEN RgX_pseudodivrem(GEN x, GEN y, GEN *ptr) compute a pseudo-quotient q and pseudo-remainder r such that $\mathtt{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$. Return q and set *ptr to r.

GEN RgX_pseudorem(GEN x, GEN y) return the remainder in the pseudo-division of x by y.

long RgX_degree(GEN x, long v) x being a t_POL and $v \geq 0$, returns the degree in v of x. Error if x is not a polynomial in v.

GEN RgXQX_pseudorem(GEN x, GEN y, GEN T) return the remainder in the pseudo-division of x by y over $R[X]/(T)$.

int ZXQX_dvd(GEN x, GEN y, GEN T) let T be a monic irreducible ZX, let x, y be t_POL whose coefficients are either t_INTs or ZX in the same variable as T. Assume further that the leading coefficient of y is an integer. Return 1 if $y|x$ in $(\mathbf{Z}[Y]/(T))[X]$, and 0 otherwise.

GEN RgXQX_pseudodivrem(GEN x, GEN y, GEN T, GEN *ptr) compute a pseudo-quotient q and pseudo-remainder r such that $\mathtt{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$ in $R[X]/(T)$. Return q and set *ptr to r.

GEN RgX_mulXn(GEN x, long n) returns $\mathtt{x} * t^n$. This may be a t_FRAC if $n < 0$ and the valuation of x is not large enough.

GEN RgX_shift(GEN x, long n) returns $\mathtt{x} * t^n$ if $n \geq 0$, and $\mathtt{x} \backslash t^{-n}$ otherwise.

GEN RgX_shift_shallow(GEN x, long n) as RgX_shift, but shallow (coefficients are not copied).

GEN RgX_rotate_shallow(GEN P, long k, long p) returns $\mathtt{P} * X^k \pmod{X^p - 1}$, assuming the degree of P is strictly less than p, and $k \geq 0$.

void RgX_shift_inplace_init(long v) $v \geq 0$, prepare for a later call to RgX_shift_inplace. Reserves v words on the stack.

GEN RgX_shift_inplace(GEN x, long v) $v \geq 0$, assume that RgX_shift_inplace_init(v) has been called (reserving v words on the stack), immediately followed by a t_POL x. Return RgX_shift(x, v) by shifting x in place. To be used as follows

```
RgX_shift_inplace_init(v);
av = avma;
...
x = gerepileupto(av, ...); /* a t_POL */
return RgX_shift_inplace(x, v);
```

long RgX_valrem(GEN P, GEN *pz) returns the valuation v of the t_POL P with respect to its main variable X. Check whether coefficients are 0 using gequal0. Set *pz to RgX_shift_shallow($P, -v$).

long RgX_val(GEN P) returns the valuation v of the t_POL P with respect to its main variable X. Check whether coefficients are 0 using gequal0.

long RgX_valrem_inexact(GEN P, GEN *z) as RgX_valrem, using isexactzero instead of gequal0.

`GEN RgX_deriv(GEN x)` returns the derivative of `x` with respect to its main variable.

`GEN RgX_integ(GEN x)` returns the primitive of `x` vanishing at 0, with respect to its main variable.

`GEN RgX_gcd(GEN x, GEN y)` returns the GCD of `x` and `y`, assumed to be `t_POL`s in the same variable.

`GEN RgX_gcd_simple(GEN x, GEN y)` as `RgX_gcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_gcd`.

`GEN RgX_extgcd(GEN x, GEN y, GEN *u, GEN *v)` returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$, and sets `*u`, `*v` to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$. Uses a generic subresultant algorithm.

`GEN RgX_extgcd_simple(GEN x, GEN y, GEN *u, GEN *v)` as `RgX_extgcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_extgcd`.

`GEN RgX_disc(GEN x)` returns the discriminant of the `t_POL` `x` with respect to its main variable.

`GEN RgX_resultant_all(GEN x, GEN y, GEN *sol)` returns `resultant(x,y)`. If `sol` is not `NULL`, sets it to the last non-constant remainder in the polynomial remainder sequence if it exists and to `gen_0` otherwise (e.g. one polynomial has degree 0). Compared to `resultant_all`, this function always uses the generic subresultant algorithm, hence always computes `sol`.

`GEN RgX_modXn_shallow(GEN x, long n)` return $\mathtt{x} \% t^n$, where $n \geq 0$. Shallow function.

`GEN RgX_modXn_eval(GEN Q, GEN x, long n)` special case of `RgX_RgXQ_eval`, when the modulus is a monomial: returns `Q(x)` modulo t^n, where $x \in R[t]$.

`GEN RgX_renormalize(GEN x)` remove leading terms in `x` which are equal to (necessarily inexact) zeros.

`GEN RgX_renormalize_lg(GEN x, long lx)` as `setlg(x, lx)` followed by `RgX_renormalize(x)`. Assumes that $\mathtt{lx} \leq \mathtt{lg(x)}$.

`GEN RgX_gtofp(GEN x, GEN prec)` returns the polynomial obtained by applying

```
gtofp(gel(x,i), prec)
```

to all coefficients of x.

`GEN RgX_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgX_gtofp(x, prec) )
```

`GEN RgX_recip(GEN P)` returns the reverse of the polynomial P, i.e. $X^{\deg P} P(1/X)$.

`GEN RgX_recip_shallow(GEN P)` shallow function of `RgX_recip`.

`GEN RgX_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q. Shallow function, not suitable for `gerepileupto`.

`long RgX_deflate_max(GEN P, long *d)` sets `d` to the largest exponent such that P is of the form $P(x^d)$ (use `gequal0` to check whether coefficients are 0), 0 if P is the zero polynomial. Returns `RgX_deflate(P,d)`.

`GEN RgX_inflate(GEN P, long d)` return $P(X^d)$. Shallow function, not suitable for `gerepileupto`.

`GEN RgX_rescale(GEN P, GEN h)` returns $h^{\deg(P)} P(x/h)$. `P` is an `RgX` and `h` is non-zero. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

`GEN RgX_unscale(GEN P, GEN h)` returns $P(hx)$. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

`GEN RgXV_unscale(GEN v, GEN h)` apply `RgX_unscale` to a vector of `RgX`.

`int RgX_is_rational(GEN P)` return 1 is the `RgX` P has only rational coefficients (`t_INT` and `t_FRAC`), and 0 otherwise.

`int RgX_is_QX(GEN P)` return 1 is the `RgX` P has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int RgX_is_ZX(GEN P)` return 1 is the `RgX` P has only `t_INT` coefficients, and 0 otherwise.

`int RgX_is_monomial(GEN x)` returns 1 (true) if `x` is a non-zero monomial in its main variable, 0 otherwise.

`long RgX_equal(GEN x, GEN y)` returns 1 if the `t_POL`s x and y have the same `degpol` and their coefficients are equal (as per `gequal`). Variable numbers are not checked. Note that this is more stringent than `gequal(x,y)`, which only checks whether $x-y$ satisfies `gequal0`; in particular, they may have different apparent degrees provided the extra leading terms are 0.

`long RgX_equal_var(GEN x, GEN y)` returns 1 if x and y have the same variable number and `RgX_equal(x,y)` is 1.

`GEN RgXQ_mul(GEN y, GEN x, GEN T)` computes $xy \bmod T$

`GEN RgXQ_sqr(GEN x, GEN T)` computes $x^2 \bmod T$

`GEN RgXQ_inv(GEN x, GEN T)` return the inverse of $x \bmod T$.

`GEN RgXQ_pow(GEN x, GEN n, GEN T)` computes $x^n \bmod T$

`GEN RgXQ_powu(GEN x, ulong n, GEN T)` computes $x^n \bmod T$, n being an `ulong`.

`GEN RgXQ_powers(GEN x, long n, GEN T)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{n}}]$ as a `t_VEC` of `RgXQ`s.

`int RgXQ_ratlift(GEN x, GEN T, long amax, long bmax, GEN *P, GEN *Q)` Assuming that `amax` + `bmax` $< \deg T$, attempts to recognize x as a rational function a/b, i.e. to find `t_POL`s P and Q such that

- $P \equiv Qx$ modulo T,
- $\deg P \leq$ `amax`, $\deg Q \leq$ `bmax`,
- $\gcd(T,P) = \gcd(P,Q)$.

If unsuccessful, the routine returns 0 and leaves P, Q unchanged; otherwise it returns 1 and sets P and Q.

`GEN RgXQ_reverse(GEN f, GEN T)` returns a `t_POL` g of degree $< n = \deg T$ such that $T(x)$ divides $(g \circ f)(x) - x$, by solving a linear system. Low-level function underlying `modreverse`: it returns a lift of [modreverse(f,T)]; faster than the high-level function since it needs not compute the characteristic polynomial of $f \bmod T$ (often already known in applications). In the trivial case where $n \leq 1$, returns a scalar, not a constant `t_POL`.

`GEN RgXQ_matrix_pow(GEN y, long n, long m, GEN P)` returns `RgXQ_powers(y,m-1,P)`, as a matrix of dimension $n \geq \deg P$.

`GEN RgXQ_norm(GEN x, GEN T)` returns the norm of `Mod(x, T)`.

`GEN RgXQ_charpoly(GEN x, GEN T, long v)` returns the characteristic polynomial of `Mod(x, T)`, in variable v.

`GEN RgX_RgXQ_eval(GEN f, GEN x, GEN T)` returns `f(x)` modulo T.

`GEN RgX_RgXQV_eval(GEN f, GEN V, GEN T)` as `RgX_RgXQ_eval(f, x, T)`, assuming V was output by `RgXQ_powers(x, n, T)` for some $n \geq 1$.

`GEN RgX_translate(GEN P, GEN c)` assume c is a scalar or a polynomials whose main variable has lower priority than the main variable X of P. Returns $P(X + c)$ (optimized for $c = \pm 1$).

`GEN RgXQX_translate(GEN P, GEN c, GEN T)` assume the main variable X of P has higher priority than the main variable Y of T and c. Return a lift of $P(X + \mathrm{Mod}(c(Y), T(Y)))$.

`GEN RgXQC_red(GEN z, GEN T)` `z` a vector whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise) in a `t_COL`.

`GEN RgXQV_red(GEN z, GEN T)` `z` a `t_POL` whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise) in a `t_VEC`.

`GEN RgXQX_red(GEN z, GEN T)` `z` a `t_POL` whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise).

`GEN RgXQX_mul(GEN x, GEN y, GEN T)`

`GEN Kronecker_to_mod(GEN z, GEN T)` $z \in R[X]$ represents an element $P(X, Y)$ in $R[X, Y]$ mod $T(Y)$ in Kronecker form, i.e. $z = P(X, X^{2*n-1})$

Let R be some commutative ring, $n = \deg T$ and let $P(X, Y) \in R[X, Y]$ lift a polynomial in $K[Y]$, where $K := R[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns the image of $P(X, t)$ in $K[t]$, with `t_POLMOD` coefficients. Not stack-clean. Note that t need not be the same variable as Y!

`GEN RgX_Rg_mul(GEN y, GEN x)` multiplies the `RgX` `y` by the scalar `x`.

`GEN RgX_muls(GEN y, long s)` multiplies the `RgX` `y` by the `long` `s`.

`GEN RgX_Rg_div(GEN y, GEN x)` divides the `RgX` `y` by the scalar `x`.

`GEN RgX_divs(GEN y, long s)` divides the `RgX` `y` by the `long` `s`.

`GEN RgX_Rg_divexact(GEN x, GEN y)` exact division of the `RgX` `y` by the scalar `x`.

`GEN RgXQX_RgXQ_mul(GEN x, GEN y, GEN T)` multiplies the `RgXQX` `y` by the scalar (`RgXQ`) `x`.

`GEN RgXQX_sqr(GEN x, GEN T)`

`GEN RgXQX_divrem(GEN x, GEN y, GEN T, GEN *pr)`

`GEN RgXQX_div(GEN x, GEN y, GEN T, GEN *r)`

`GEN RgXQX_rem(GEN x, GEN y, GEN T, GEN *r)`

Chapter 8:
Operations on general PARI objects

8.1 Assignment.

It is in general easier to use a direct conversion, e.g. `y = stoi(s)`, than to allocate a target of correct type and sufficient size, then assign to it:

```
GEN y = cgeti(3); affsi(s, y);
```

These functions can still be moderately useful in complicated garbage collecting scenarios but you will be better off not using them.

`void gaffsg(long s, GEN x)` assigns the `long s` into the object `x`.

`void gaffect(GEN x, GEN y)` assigns the object `x` into the object `y`. Both `x` and `y` must be scalar types. Type conversions (e.g. from `t_INT` to `t_REAL` or `t_INTMOD`) occur if legitimate.

`int is_universal_constant(GEN x)` returns 1 if x is a global PARI constant you should never assign to (such as `gen_1`), and 0 otherwise.

8.2 Conversions.

8.2.1 Scalars.

`double rtodbl(GEN x)` applied to a `t_REAL x`, converts `x` into a `double` if possible.

`GEN dbltor(double x)` converts the `double x` into a `t_REAL`.

`long dblexpo(double x)` returns `expo(dbltor(x))`, but faster and without cluttering the stack.

`ulong dblmantissa(double x)` returns the most significant word in the mantissa of `dbltor(x)`.

`double gtodouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), converts `x` into a `double` if possible.

`long gtos(GEN x)` converts the `t_INT x` to a small integer if possible, otherwise raise an exception. This function is similar to `itos`, slightly slower since it checks the type of `x`.

`double dbllog2r(GEN x)` assuming `x` is a non-zero `t_REAL`, returns an approximation to `log2(|x|)`.

`long gtolong(GEN x)` if `x` is an integer (not necessarily a `t_INT`), converts `x` into a `long` if possible.

`GEN fractor(GEN x, long l)` applied to a `t_FRAC x`, converts `x` into a `t_REAL` of length `prec`.

`GEN quadtofp(GEN x, long l)` applied to a `t_QUAD x`, converts `x` into a `t_REAL` or `t_COMPLEX` depending on the sign of the discriminant of `x`, to precision `l BITS_IN_LONG`-bit words.

`GEN cxtofp(GEN x, long prec)` converts the `t_COMPLEX x` to a a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`.

GEN cxcompotor(GEN x, long prec) converts the t_INT, t_REAL or t_FRAC x to a t_REAL of length prec. These are all the real types which may occur as components of a t_COMPLEX; special case of gtofp (introduced so that the latter is not recursive and can thus be inlined).

GEN gtofp(GEN x, long prec) converts the complex number x (t_INT, t_REAL, t_FRAC, t_QUAD or t_COMPLEX) to either a t_REAL or t_COMPLEX whose components are t_REAL of precision prec; not necessarily of *length* prec: a real 0 may be given as real_0(...)). If the result is a t_COMPLEX extra care is taken so that its modulus really has accuracy prec: there is a problem if the real part of the input is an exact 0; indeed, converting it to real_0(prec) would be wrong if the imaginary part is tiny, since the modulus would then become equal to 0, as in $1.E-100+0.E-28 = 0.E-28$.

GEN gtomp(GEN z, long prec) converts the real number x (t_INT, t_REAL, t_FRAC, real t_QUAD) to either a t_INT or a t_REAL of precision prec. Not memory clean if x is a t_INT: we return x itself and not a copy.

GEN gcvtop(GEN x, GEN p, long l) converts x into a t_PADIC of precision l. Works componentwise on recursive objects, e.g. t_POL or t_VEC. Converting 0 yields $O(p^l)$; converting a non-zero number yield a result well defined modulo $p^{v_p(x)+l}$.

GEN cvtop(GEN x, GEN p, long l) as gcvtop, assuming that x is a scalar.

GEN cvtop2(GEN x, GEN y) y being a p-adic, converts the scalar x to a p-adic of the same accuracy. Shallow function.

GEN cvstop2(long s, GEN y) y being a p-adic, converts the scalar s to a p-adic of the same accuracy. Shallow function.

GEN gprec(GEN x, long l) returns a copy of x whose precision is changed to l digits. The precision change is done recursively on all components of x. Digits means *decimal*, p-adic and X-adic digits for t_REAL, t_SER, t_PADIC components, respectively.

GEN gprec_w(GEN x, long l) returns a shallow copy of x whose t_REAL components have their precision changed to l *words*. This is often more useful than gprec.

GEN gprec_wtrunc(GEN x, long l) returns a shallow copy of x whose t_REAL components have their precision *truncated* to l *words*. Contrary to gprec_w, this function may never increase the precision of x.

8.2.2 Modular objects / lifts.

GEN gmodulo(GEN x, GEN y) creates the object **Mod**(x,y) on the PARI stack, where x and y are either both t_INTs, and the result is a t_INTMOD, or x is a scalar or a t_POL and y a t_POL, and the result is a t_POLMOD.

GEN gmodulgs(GEN x, long y) same as **gmodulo** except y is a long.

GEN gmodulsg(long x, GEN y) same as **gmodulo** except x is a long.

GEN gmodulss(long x, long y) same as **gmodulo** except both x and y are longs.

GEN liftall_shallow(GEN x) shallow version of liftall

GEN liftint_shallow(GEN x) shallow version of liftint

GEN liftpol_shallow(GEN x) shallow version of liftpol

GEN centerlift0(GEN x,long v) DEPRECATED, kept for backward compatibility only: use either lift0(x, v) or centerlift(x).

8.2.3 Between polynomials and coefficient arrays.

`GEN gtopoly(GEN x, long v)` converts or truncates the object `x` into a `t_POL` with main variable number `v`. A common application would be the conversion of coefficient vectors (coefficients are given by decreasing degree). E.g. `[2,3]` goes to `2*v + 3`

`GEN gtopolyrev(GEN x, long v)` converts or truncates the object `x` into a `t_POL` with main variable number `v`, but vectors are converted in reverse order compared to `gtopoly` (coefficients are given by increasing degree). E.g. `[2,3]` goes to `3*v + 2`. In other words the vector represents a polynomial in the basis $(1, v, v^2, v^3, \ldots)$.

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. For internal use. Normalizing means deleting all leading *exact* zeroes (as per `isexactzero`), except if the polynomial turns out to be 0, in which case we try to find a coefficient c which is a non-rational zero, and return the constant polynomial c. (We do this so that information about the base ring is not lost.)

`GEN normalizepol_lg(GEN x, long l)` applies `normalizepol` to `x`, pretending that `lg(x)` is l, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `normalizepol(x)`.

`GEN normalizepol_approx(GEN x, long lx)` as `normalizepol_lg`, with the difference that we just delete all leading zeroes (as per `gequal0`). This rougher normalization is used when we have no other choice, for instance before attempting a Euclidean division by x.

The following routines do *not* copy coefficients on the stack (they only move pointers around), hence are very fast but not suitable for `gerepile` calls. Recall that an `RgV` (resp. an `RgX`, resp. an `RgM`) is a `t_VEC` or `t_COL` (resp. a `t_POL`, resp. a `t_MAT`) with arbitrary components. Similarly, an `RgXV` is a `t_VEC` or `t_COL` with `RgX` components, etc.

`GEN RgV_to_RgX(GEN x, long v)` converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopolyrev`, without copy).

`GEN RgV_to_RgX_reverse(GEN x, long v)` converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopoly`, without copy).

`GEN RgX_to_RgV(GEN x, long N)` converts the `t_POL` `x` to a `t_COL` `v` with `N` components. Coefficients of `x` are listed by increasing degree, so that `y[i]` is the coefficient of the term of degree $i-1$ in `x`.

`GEN Rg_to_RgV(GEN x, long N)` as `RgX_to_RgV`, except that other types than `t_POL` are allowed for `x`, which is then considered as a constant polynomial.

`GEN RgM_to_RgXV(GEN x, long v)` converts the `RgM` `x` to a `t_VEC` of `RgX`, by repeated calls to `RgV_to_RgX`.

`GEN RgV_to_RgM(GEN v, long N)` converts the vector `v` to a `t_MAT` with `N` rows, by repeated calls to `Rg_to_RgV`.

`GEN RgXV_to_RgM(GEN v, long N)` converts the vector of `RgX` `v` to a `t_MAT` with `N` rows, by repeated calls to `RgX_to_RgV`.

`GEN RgM_to_RgXX(GEN x, long v,long w)` converts the `RgM` `x` into a `t_POL` in variable `v`, whose coefficients are `t_POL`s in variable `w`. This is a shortcut for

```
RgV_to_RgX( RgM_to_RgXV(x, w), v );
```

There are no consistency checks with respect to variable priorities: the above is an invalid object if `varncmp(v, w)` ≥ 0.

`GEN RgXX_to_RgM(GEN x, long N)` converts the `t_POL x` with `RgX` (or constant) coefficients to a matrix with `N` rows.

`GEN RgXY_swap(GEN P, long n, long w)` converts the bivariate polynomial `P`(u, v) (a `t_POL` with `t_POL` or scalar coefficients) to $P($`pol_x`$[$`w`$], u)$, assuming `n` is an upper bound for $\deg_v($`P`$)$.

`GEN RgXY_swapspec(GEN C, long n, long w, long lP)` as `RgXY_swap` where the coefficients of P are given by `gel(C,0)`,...,`gel(C,lP-1)`.

`GEN RgX_to_ser(GEN x, long l)` applied to a `t_POL x`, creates a *shallow* `t_SER` of length $l \geq 2$ starting with `x`. Unless the polynomial is an exact zero, the coefficient of lowest degree T^d of the result is not an exact zero (as per `isexactzero`). The remainder is $O(T^{d+l})$.

`GEN RgX_to_ser_inexact(GEN x, long l)` applied to a `t_POL x`, creates a *shallow* `t_SER` of length `l` starting with `x`. Unless the polynomial is zero, the coefficient of lowest degree T^d of the result is not zero (as per `gequal0`). The remainder is $O(T^{d+l})$.

`GEN rfrac_to_ser(GEN x, long l)` applied to a `t_RFRAC x`, creates a `t_SER` of length `l` congruent to x. Not memory-clean but suitable for `gerepileupto`.

`GEN gtoser(GEN s, long v, long d)` converts the object s into a `t_SER` with main variable number `v` and $d > 0$ significant terms. More precisely

- if s is a scalar, we return a constant power series with d significant terms.
- if s is a `t_POL`, it is truncated to d terms if needed.
- If s is a vector, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`), and the precision d is *ignored*.
- If s is already a power series in v, we retur a copy, and the precision d is again *ignored*.

`GEN gtocol(GEN x)` converts the object `x` into a `t_COL`

`GEN gtomat(GEN x)` converts the object `x` into a `t_MAT`.

`GEN gtovec(GEN x)` converts the object `x` into a `t_VEC`.

`GEN gtovecsmall(GEN x)` converts the object `x` into a `t_VECSMALL`.

`GEN normalize(GEN x)` applied to an unnormalized `t_SER x` (i.e. type `t_SER` with all coefficients correctly set except that `x[2]` might be zero), normalizes `x` correctly in place. Returns `x`. For internal use.

`GEN serchop0(GEN s)` given a `t_SER` of the form $x^v s(x)$, with $s(0) \neq 0$, return $x^v(s - s(0))$. Shallow function.

8.3 Constructors.

8.3.1 Clean constructors.

`GEN zeropadic(GEN p, long n)` creates a 0 `t_PADIC` equal to $O(\mathtt{p}^{\mathtt{n}})$.

`GEN zeroser(long v, long n)` creates a 0 `t_SER` in variable `v` equal to $O(X^{\mathtt{n}})$.

`GEN scalarser(GEN x, long v, long prec)` creates a constant `t_SER` in variable `v` and precision `prec`, whose constant coefficient is (a copy of) `x`, in other words $\mathtt{x} + O(\mathtt{v}^{\mathtt{prec}})$. Assumes that `x` is non-zero.

`GEN pol_0(long v)` Returns the constant polynomial 0 in variable v.

`GEN pol_1(long v)` Returns the constant polynomial 1 in variable v.

`GEN pol_x(long v)` Returns the monomial of degree 1 in variable v.

`GEN pol_x_powers(long N, long v)` returns the powers of `pol_x(v)`, of degree 0 to N, in a vector with $N + 1$ components.

`GEN scalarpol(GEN x, long v)` creates a constant `t_POL` in variable `v`, whose constant coefficient is (a copy of) `x`.

`GEN deg1pol(GEN a, GEN b,long v)` creates the degree 1 `t_POL` a`pol_x`$(v) + b$

`GEN zeropol(long v)` is identical `pol_0`.

`GEN zerocol(long n)` creates a `t_COL` with `n` components set to `gen_0`.

`GEN zerovec(long n)` creates a `t_VEC` with `n` components set to `gen_0`.

`GEN col_ei(long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).

`GEN vec_ei(long n, long i)` creates a `t_VEC` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).

`GEN trivial_fact(void)` returns the trivial (empty) factorization `Mat([]~,[]~)`

`GEN prime_fact(GEN x)` returns the factorization `Mat([x]~, [1]~)`

`GEN Rg_col_ei(GEN x, long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `x`.

`GEN vecsmall_ei(long n, long i)` creates a `t_VECSMALL` with `n` components set to `0`, but for the `i`-th one which is set to `1` (`i`-th vector in the canonical basis).

`GEN scalarcol(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a copy of `x`. (The name comes from `RgV_isscalar`.)

`GEN mkintmodu(ulong x, ulong y)` creates the `t_INTMOD Mod(x, y)`. The inputs must satisfy $x < y$.

`GEN zeromat(long m, long n)` creates a `t_MAT` with `m` x `n` components set to `gen_0`. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns. To fully allocate a matrix initialized with zero entries, use `zeromatcopy`.

`GEN zeromatcopy(long m, long n)` creates a `t_MAT` with `m` x `n` components set to `gen_0`.

`GEN matid(long n)` identity matrix in dimension `n` (with components `gen_1` and`gen_0`).

`GEN scalarmat(GEN x, long n)` scalar matrix, `x` times the identity.

`GEN scalarmat_s(long x, long n)` scalar matrix, `stoi(x)` times the identity.

`GEN vecrange(GEN a, GEN b)` returns the `t_VEC` $[a..b]$.

`GEN vecrangess(long a, long b)` returns the `t_VEC` $[a..b]$.

See also next section for analogs of the following functions:

`GEN mkfraccopy(GEN x, GEN y)` creates the `t_FRAC` x/y. Assumes that $y > 1$ and $(x, y) = 1$.

`GEN mkrfraccopy(GEN x, GEN y)` creates the `t_RFRAC` x/y. Assumes that y is a `t_POL`, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

`GEN mkcolcopy(GEN x)` creates a 1-dimensional `t_COL` containing `x`.

`GEN mkmatcopy(GEN x)` creates a 1-by-1 `t_MAT` wrapping the `t_COL x`.

`GEN mkveccopy(GEN x)` creates a 1-dimensional `t_VEC` containing `x`.

`GEN mkvec2copy(GEN x, GEN y)` creates a 2-dimensional `t_VEC` equal to `[x,y]`.

`GEN mkcols(long x)` creates a 1-dimensional `t_COL` containing `stoi(x)`.

`GEN mkcol2s(long x, long y)` creates a 2-dimensional `t_COL` containing `[stoi(x), stoi(y)]` .

`GEN mkcol3s(long x, long y, long z)` creates a 3-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z)]` .

`GEN mkcol4s(long x, long y, long z, long t)` creates a 4-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]` .

`GEN mkvecs(long x)` creates a 1-dimensional `t_VEC` containing `stoi(x)`.

`GEN mkvec2s(long x, long y)` creates a 2-dimensional `t_VEC` containing `[stoi(x), stoi(y)]`.

`GEN mkvec3s(long x, long y, long z)` creates a 3-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z)]`.

`GEN mkvec4s(long x, long y, long z, long t)` creates a 4-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]`.

`GEN mkvecsmall(long x)` creates a 1-dimensional `t_VECSMALL` containing `x`.

`GEN mkvecsmall2(long x, long y)` creates a 2-dimensional `t_VECSMALL` containing `[x, y]`.

`GEN mkvecsmall3(long x, long y, long z)` creates a 3-dimensional `t_VECSMALL` containing `[x, y, z]`.

`GEN mkvecsmall4(long x, long y, long z, long t)` creates a 4-dimensional `t_VECSMALL` containing `[x, y, z, t]`.

`GEN mkvecsmalln(long n, ...)` returns the `t_VECSMALL` whose n coefficients (`long`) follow.

8.3.2 Unclean constructors.

Contrary to the policy of general PARI functions, the functions in this subsection do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. In particular, they are faster than their clean equivalent (which may not exist). *If* you restrict their arguments to universal objects (e.g `gen_0`), then the above warning does not apply.

`GEN mkcomplex(GEN x, GEN y)` creates the `t_COMPLEX` $x + iy$.

`GEN mulcxI(GEN x)` creates the `t_COMPLEX` ix. The result in general contains data pointing back to the original x. Use `gcopy` if this is a problem. But in most cases, the result is to be used immediately, before x is subject to garbage collection.

`GEN mulcxmI(GEN x)`, as `mulcxI`, but returns the `t_COMPLEX` $-ix$.

`GEN mkquad(GEN n, GEN x, GEN y)` creates the `t_QUAD` $x + yw$, where w is a root of n, which is of the form `quadpoly(D)`.

`GEN mkfrac(GEN x, GEN y)` creates the `t_FRAC` x/y. Assumes that $y > 1$ and $(x, y) = 1$.

`GEN mkrfrac(GEN x, GEN y)` creates the `t_RFRAC` x/y. Assumes that y is a `t_POL`, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

`GEN mkcol(GEN x)` creates a 1-dimensional `t_COL` containing `x`.

`GEN mkcol2(GEN x, GEN y)` creates a 2-dimensional `t_COL` equal to `[x,y]`.

`GEN mkcol3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_COL` equal to `[x,y,z]`.

`GEN mkcol4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_COL` equal to `[x,y,z,t]`.

`GEN mkcol5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_COL` equal to $[a_1, a_2, a_3, a_4, a_5]$.

`GEN mkintmod(GEN x, GEN y)` creates the `t_INTMOD` `Mod(x, y)`. The inputs must be `t_INT`s satisfying $0 \leq x < y$.

`GEN mkpolmod(GEN x, GEN y)` creates the `t_POLMOD` `Mod(x, y)`. The input must satisfy $\deg x < \deg y$ with respect to the main variable of the `t_POL` y. x may be a scalar.

`GEN mkmat(GEN x)` creates a 1-column `t_MAT` with column x (a `t_COL`).

`GEN mkmat2(GEN x, GEN y)` creates a 2-column `t_MAT` with columns x, y (`t_COL`s of the same length).

`GEN mkmat3(GEN x, GEN y, GEN z)` creates a 3-column `t_MAT` with columns x, y, z (`t_COL`s of the same length).

`GEN mkmat4(GEN x, GEN y, GEN z, GEN t)` creates a 4-column `t_MAT` with columns x, y, z, t (`t_COL`s of the same length).

`GEN mkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` creates a 5-column `t_MAT` with columns x, y, z, t, u (`t_COL`s of the same length).

`GEN mkvec(GEN x)` creates a 1-dimensional `t_VEC` containing `x`.

`GEN mkvec2(GEN x, GEN y)` creates a 2-dimensional `t_VEC` equal to `[x,y]`.

`GEN mkvec3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_VEC` equal to `[x,y,z]`.

`GEN mkvec4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_VEC` equal to `[x,y,z,t]`.

`GEN mkvec5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_VEC` equal to $[a_1, a_2, a_3, a_4, a_5]$.

`GEN mkqfi(GEN x, GEN y, GEN z)` creates `t_QFI` equal to `Qfb(x,y,z)`, assuming that $y^2 - 4xz < 0$.

`GEN mkerr(long n)` returns a `t_ERROR` with error code n (`enum err_list`).

It is sometimes useful to return such a container whose entries are not universal objects, but nonetheless suitable for `gerepileupto`. If the entries can be computed at the time the result is returned, the following macros achieve this effect:

`GEN retmkvec(GEN x)` returns a vector containing the single entry x, where the vector root is created just before the function argument x is evaluated. Expands to

```
{
  GEN res = cgetg(2, t_VEC);
  gel(res, 1) = x; /*  or rather, the expansion of x */
  return res;
}
```

For instance, the `retmkvec(gcopy(x))` returns a clean object, just like `return mkveccopy(x)` would.

`GEN retmkvec2(GEN x, GEN y)` returns the 2-dimensional `t_VEC [x,y]`.

`GEN retmkvec3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_VEC [x,y,z]`.

`GEN retmkvec4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_VEC [x,y,z,t]`.

`GEN retmkvec5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional row vector `[x,y,z,t,u]`.

`GEN retconst_vec(long n, GEN x)` returns the n-dimensional `t_VEC` whose entries are constant and all equal to x.

`GEN retmkcol(GEN x)` returns the 1-dimensional `t_COL [x]` .

`GEN retmkcol2(GEN x, GEN y)` returns the 2-dimensional `t_COL [x,y]` .

`GEN retmkcol3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_COL [x,y,z]` .

`GEN retmkcol4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_COL [x,y,z,t]` .

`GEN retmkcol5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional column vector `[x,y,z,t,u]` .

`GEN retconst_col(long n, GEN x)` returns the n-dimensional `t_COL` whose entries are constant and all equal to x.

`GEN retmkmat(GEN x)` returns the 1-column `t_MAT` with colum `x`.

`GEN retmkmat2(GEN x, GEN y)` returns the 2-column `t_MAT` with columns `x`, `y`.

`GEN retmkmat3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_MAT` with columns `x`, `y`, `z`.

`GEN retmkmat4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`.

`GEN retmkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`, `u`.

`GEN retmkcomplex(GEN x, GEN y)` returns the `t_COMPLEX` `x + I*y`.

`GEN retmkfrac(GEN x, GEN y)` returns the `t_FRAC` `x / y`. Assume x and y are coprime and $y > 1$.

`GEN retmkintmod(GEN x, GEN y)` returns the `t_INTMOD` `Mod(x, y)`.

`GEN retmkqfi(GEN a, GEN b, GEN c)`.

`GEN retmkqfr(GEN a, GEN b, GEN c, GEN d)`.

`GEN retmkquad(GEN n, GEN a, GEN b)`.

`GEN retmkpolmod(GEN x, GEN y)` returns the `t_POLMOD` `Mod(x, y)`.

`GEN mkintn(long n, ...)` returns the non-negative `t_INT` whose development in base 2^{32} is given by the following n words (`unsigned long`). It is assumed that all such arguments are less than 2^{32} (the actual `sizeof(long)` is irrelevant, the behavior is also as above on 64-bit machines).

```
mkintn(3, a2, a1, a0);
```

returns $a_2 2^{64} + a_1 2^{32} + a_0$.

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose n coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

`GEN scalarcol_shallow(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a shallow copy of `x`. (The name comes from `RgV_isscalar`.)

`GEN scalarmat_shallow(GEN x, long n)` creates an $n \times n$ scalar matrix whose diagonal is set to shallow copies of the scalar `x`.

`GEN diagonal_shallow(GEN x)` returns a diagonal matrix whose diagonal is given by the vector x. Shallow function.

`GEN scalarpol_shallow(GEN a, long v)` returns the degree 0 `t_POL` a`pol_x`$(v)^0$.

`GEN deg1pol_shallow(GEN a, GEN b,long v)` returns the degree 1 `t_POL` a`pol_x`$(v) + b$

`GEN zeropadic_shallow(GEN p, long n)` returns a (shallow) 0 `t_PADIC` equal to $O($`p`$^{\mathtt{n}})$.

8.3.3 From roots to polynomials.

`GEN deg1_from_roots(GEN L, long v)` given a vector L of scalars, returns the vector of monic linear polynomials in variable v whose roots are the $L[i]$, i.e. the $x - L[i]$.

`GEN roots_from_deg1(GEN L)` given a vector L of monic linear polynomials, return their roots, i.e. the $-L[i](0)$.

`GEN roots_to_pol(GEN L, long v)` given a vector of scalars L, returns the monic polynomial in variable v whose roots are the $L[i]$. Calls `divide_conquer_prod`, so leaves some garbage on stack, but suitable for `gerepileupto`.

`GEN roots_to_pol_r1(GEN L, long v, long r1)` as `roots_to_pol` assuming the first r_1 roots are "real", and the following ones are representatives of conjugate pairs of "complex" roots. So if L has $r_1 + r_2$ elements, we obtain a polynomial of degree $r_1 + 2r_2$. In most applications, the roots are indeed real and complex, but the implementation assumes only that each "complex" root z introduces a quadratic factor $X^2 - \texttt{trace}(z)X + \texttt{norm}(z)$. Calls `divide_conquer_prod`. Calls `divide_conquer_prod`, so leaves some garbage on stack, but suitable for `gerepileupto`.

8.4 Integer parts.

`GEN gfloor(GEN x)` creates the floor of `x`, i.e. the (true) integral part.

`GEN gfrac(GEN x)` creates the fractional part of `x`, i.e. `x` minus the floor of `x`.

`GEN gceil(GEN x)` creates the ceiling of `x`.

`GEN ground(GEN x)` rounds towards $+\infty$ the components of `x` to the nearest integers.

`GEN grndtoi(GEN x, long *e)` same as `ground`, but in addition sets `*e` to the binary exponent of $x - \texttt{ground}(x)$. If this is positive, all significant bits are lost. This kind of situation raises an error message in **ground** but not in **grndtoi**.

`GEN gtrunc(GEN x)` truncates `x`. This is the false integer part if `x` is a real number (i.e. the unique integer closest to `x` among those between 0 and `x`). If `x` is a `t_SER`, it is truncated to a `t_POL`; if `x` is a `t_RFRAC`, this takes the polynomial part.

`GEN gtrunc2n(GEN x, long n)` creates the floor of 2^n`x`, this is only implemented for `t_INT`, `t_REAL`, `t_FRAC` and `t_COMPLEX` of those.

`GEN gcvtoi(GEN x, long *e)` analogous to **grndtoi** for `t_REAL` inputs except that rounding is replaced by truncation. Also applies componentwise for vector or matrix inputs; otherwise, sets `*e` to `-HIGHEXPOBIT` (infinite real accuracy) and return `gtrunc(x)`.

8.5 Valuation and shift.

`GEN gshift[z](GEN x, long n[, GEN z])` yields the result of shifting (the components of) `x` left by `n` (if `n` is non-negative) or right by $-$`n` (if `n` is negative). Applies only to `t_INT` and vectors/matrices of such. For other types, it is simply multiplication by $2^{\texttt{n}}$.

`GEN gmul2n[z](GEN x, long n[, GEN z])` yields the product of `x` and $2^{\texttt{n}}$. This is different from `gshift` when `n` is negative and `x` is a `t_INT`: **gshift** truncates, while **gmul2n** creates a fraction if necessary.

`long gvaluation(GEN x, GEN p)` returns the greatest exponent e such that $\texttt{p}^e$ divides `x`, when this makes sense.

`long gval(GEN x, long v)` returns the highest power of the variable number `v` dividing the `t_POL` `x`.

8.6 Comparison operators.

8.6.1 Generic.

`long gcmp(GEN x, GEN y)` comparison of `x` with `y`: returns 1 ($x > y$), 0 ($x = y$) or -1 ($x < y$). Two `t_STR` are compared using the standard lexicographic ordering; a `t_STR` is considered strictly larger than any non-string type. If neither x nor y is a `t_STR`, their allowed types are `t_INT`, `t_REAL` or `t_FRAC`. Used `cmp_universal` to compare arbitrary `GEN`s.

`long lexcmp(GEN x, GEN y)` comparison of `x` with `y` for the lexicographic ordering; when comparing objects of different lengths whose components are all equal up to the smallest of their length, consider that the longest is largest. Consider scalars as 1-component vectors. Return `gcmp`(x, y) if both arguments are scalars.

`int gequalX(GEN x)` return 1 (true) if `x` is a variable (monomial of degree 1 with `t_INT` coefficients equal to 1 and 0), and 0 otherwise

`long gequal(GEN x, GEN y)` returns 1 (true) if `x` is equal to `y`, 0 otherwise. A priori, this makes sense only if `x` and `y` have the same type, in which case they are recursively compared componentwise. When the types are different, a `true` result means that `x - y` was successfully computed and that `gequal0` found it equal to 0. In particular

```
gequal(cgetg(1, t_VEC), gen_0)
```

is true, and the relation is not transitive. E.g. an empty `t_COL` and an empty `t_VEC` are not equal but are both equal to `gen_0`.

`long gidentical(GEN x, GEN y)` returns 1 (true) if `x` is identical to `y`, 0 otherwise. In particular, the types and length of `x` and `y` must be equal. This test is much stricter than `gequal`, in particular, `t_REAL` with different accuracies are tested different. This relation is transitive.

8.6.2 Comparison with a small integer.

`int isexactzero(GEN x)` returns 1 (true) if `x` is exactly equal to 0 (including `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. This includes recursive objects, for instance vectors, whose components are 0.

`int isrationalzero(GEN x)` returns 1 (true) if `x` is equal to an integer 0 (excluding `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. Contrary to `isintzero`, this includes recursive objects, for instance vectors, whose components are 0.

`int ismpzero(GEN x)` returns 1 (true) if `x` is a `t_INT` or a `t_REAL` equal to 0.

`int isintzero(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 0.

`int isint1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 1.

`int isintm1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to -1.

`int equali1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to 1, and return 0 (false) otherwise.

`int equalim1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to -1, and return 0 (false) otherwise.

`int is_pm1(GEN x)`. Assuming that `x` is a *non-zero* `t_INT`, return 1 (true) if `x` is equal to -1 or 1, and return 0 (false) otherwise.

`int gequal0(GEN x)` returns 1 (true) if `x` is equal to 0, 0 (false) otherwise.

`int gequal1(GEN x)` returns 1 (true) if `x` is equal to 1, 0 (false) otherwise.

`int gequalm1(GEN x)` returns 1 (true) if `x` is equal to -1, 0 (false) otherwise.

`long gcmpsg(long s, GEN x)`

`long gcmpgs(GEN x, long s)` comparison of `x` with the `long s`.

`GEN gmaxsg(long s, GEN x)`

`GEN gmaxgs(GEN x, long s)` returns the largest of `x` and the `long s` (converted to `GEN`)

`GEN gminsg(long s, GEN x)`

`GEN gmings(GEN x, long s)` returns the smallest of `x` and the `long s` (converted to `GEN`)

`long gequalsg(long s, GEN x)`

`long gequalgs(GEN x, long s)` returns 1 (true) if `x` is equal to the `long s`, 0 otherwise.

8.7 Miscellaneous Boolean functions.

`int isrationalzeroscalar(GEN x)` equivalent to, but faster than,

```
is_scalar_t(typ(x)) && isrationalzero(x)
```

`int isinexact(GEN x)` returns 1 (true) if x has an inexact component, and 0 (false) otherwise.

`int isinexactreal(GEN x)` return 1 if x has an inexact `t_REAL` component, and 0 otherwise.

`int isrealappr(GEN x, long e)` applies (recursively) to complex inputs; returns 1 if x is approximately real to the bit accuracy e, and 0 otherwise. This means that any `t_COMPLEX` component must have imaginary part t satisfying $\texttt{gexpo}(t) < e$.

`int isint(GEN x, GEN *n)` returns 0 (false) if `x` does not round to an integer. Otherwise, returns 1 (true) and set `n` to the rounded value.

`int issmall(GEN x, long *n)` returns 0 (false) if `x` does not round to a small integer (suitable for `itos`). Otherwise, returns 1 (true) and set `n` to the rounded value.

`long iscomplex(GEN x)` returns 1 (true) if `x` is a complex number (of component types embeddable into the reals) but is not itself real, 0 if `x` is a real (not necessarily of type `t_REAL`), or raises an error if `x` is not embeddable into the complex numbers.

8.7.1 Obsolete.

The following less convenient comparison functions and Boolean operators were used by the historical GP interpreter. They are provided for backward compatibility only and should not be used:

```
GEN gle(GEN x, GEN y)

GEN glt(GEN x, GEN y)

GEN gge(GEN x, GEN y)

GEN ggt(GEN x, GEN y)

GEN geq(GEN x, GEN y)

GEN gne(GEN x, GEN y)

GEN gor(GEN x, GEN y)

GEN gand(GEN x, GEN y)

GEN gnot(GEN x, GEN y)
```

8.8 Sorting.

8.8.1 Basic sort.

`GEN sort(GEN x)` sorts the vector `x` in ascending order using a mergesort algorithm, and `gcmp` as the underlying comparison routine (returns the sorted vector). This routine copies all components of x, use `gen_sort_inplace` for a more memory-efficient function.

`GEN lexsort(GEN x)`, as `sort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN vecsort(GEN x, GEN k)`, as `sort`, but sorts the vector `x` in ascending *lexicographic* order, according to the entries of the `t_VECSMALL k`. For example, if $\texttt{k} = [2, 1, 3]$, sorting will be done with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

8.8.2 Indirect sorting.

`GEN indexsort(GEN x)` as `sort`, but only returns the permutation which, applied to `x`, would sort the vector. The result is a `t_VECSMALL`.

`GEN indexlexsort(GEN x)`, as `indexsort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN indexvecsort(GEN x, GEN k)`, as `vecsort`, but only returns the permutation that would sort the vector `x`.

`long vecindexmin(GEN x)` returns the index for a maximal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)` returns the index for a maximal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)`

8.8.3 Generic sort and search. The following routines allow to use an arbitrary comparison function `int (*cmp)(void* data, GEN x, GEN y)`, such that `cmp(data,x,y)` returns a negative result if $x < y$, a positive one if $x > y$ and 0 if $x = y$. The `data` argument is there in case your `cmp` requires additional context.

`GEN gen_sort(GEN x, void *data, int (*cmp)(void *,GEN,GEN))`, as `sort`, with an explicit comparison routine.

`GEN gen_sort_uniq(GEN x, void *data, int (*cmp)(void *,GEN,GEN))`, as `gen_sort`, removing duplicate entries.

`GEN gen_indexsort(GEN x, void *data, int (*cmp)(void*,GEN,GEN))`, as `indexsort`.

`GEN gen_indexsort_uniq(GEN x, void *data, int (*cmp)(void*,GEN,GEN))`, as `indexsort`, removing duplicate entries.

`void gen_sort_inplace(GEN x, void *data, int (*cmp)(void*,GEN,GEN), GEN *perm)` sort `x` in place, without copying its components. If `perm` is non-`NULL`, it is set to the permutation that would sort the original `x`.

`GEN gen_setminus(GEN A, GEN B, int (*cmp)(GEN,GEN))` given two sorted vectors A and B, returns the vector of elements of A not belonging to B.

`GEN sort_factor(GEN y, void *data, int (*cmp)(void *,GEN,GEN))`: assuming `y` is a factorization matrix, sorts its rows in place (no copy is made) according to the comparison function `cmp` applied to its first column.

`GEN merge_sort_uniq(GEN x,GEN y, void *data, int (*cmp)(void *,GEN,GEN))` assuming `x` and `y` are sorted vectors, with respect to the `cmp` comparison function, return a sorted concatenation, with duplicates removed.

`GEN merge_factor(GEN fx, GEN fy, void *data, int (*cmp)(void *,GEN,GEN))` let `fx` and `fy` be factorization matrices for X and Y sorted with respect to the comparison function `cmp` (see `sort_factor`), returns the factorization of $X * Y$. Zero exponents in the latter factorization are preserved, e.g. when merging the factorization of 2 and $1/2$, the result is 2^0.

`long gen_search(GEN v, GEN y, long flag, void *data, int (*cmp)(void*,GEN,GEN))`. Let `v` be a vector sorted according to `cmp(data,a,b)`; look for an index i such that `v[`i`]` is equal to `y`. `flag` has the same meaning as in `setsearch`: if `flag` is 0, return i if it exists and 0 otherwise; if `flag` is non-zero, return 0 if i exists and the index where `y` should be inserted otherwise.

`long tablesearch(GEN T, GEN x, int (*cmp)(GEN,GEN))` is a faster implementation for the common case `gen_search(T,x,0,cmp,cmp_nodata)`.

8.8.4 Further useful comparison functions.

`int cmp_universal(GEN x, GEN y)` a somewhat arbitrary universal comparison function, devoid of sensible mathematical meaning. It is transitive, and returns 0 if and only if `gidentical(x,y)` is true. Useful to sort and search vectors of arbitrary data.

`int cmp_nodata(void *data, GEN x, GEN y)`. This function is a hack used to pass an existing basic comparison function lacking the `data` argument, i.e. with prototype `int (*cmp)(GEN x, GEN y)`. Instead of `gen_sort(x, NULL, cmp)` which may or may not work depending on how your compiler handles typecasts between incompatible function pointers, one should use `gen_sort(x, (void*)cmp, cmp_nodata)`.

Here are a few basic comparison functions, to be used with `cmp_nodata`:

`int ZV_cmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order).

`int cmp_RgX(GEN x, GEN y)` compare two polynomials, which we assume have the same main variable (lexicographic order). The coefficients are compared using `gcmp`.

`int cmp_prime_over_p(GEN x, GEN y)` compare two prime ideals, which we assume divide the same prime number. The comparison is ad hoc but orders according to increasing residue degrees.

`int cmp_prime_ideal(GEN x, GEN y)` compare two prime ideals in the same nf. Orders by increasing primes, breaking ties using `cmp_prime_over_p`.

Finally a more elaborate comparison function:

`int gen_cmp_RgX(void *data, GEN x, GEN y)` compare two polynomials, ordering first by increasing degree, then according to the coefficient comparison function:

```
int (*cmp_coeff)(GEN,GEN) = (int(*)(GEN,GEN)) data;
```

8.9 Divisibility, Euclidean division.

`GEN gdivexact(GEN x, GEN y)` returns the quotient `x/y`, assuming `y` divides `x`. Not stack clean if $y = 1$ (we return x, not a copy).

`int gdvd(GEN x, GEN y)` returns 1 (true) if `y` divides `x`, 0 otherwise.

`GEN gdiventres(GEN x, GEN y)` creates a 2-component vertical vector whose components are the true Euclidean quotient and remainder of `x` and `y`.

`GEN gdivent[z](GEN x, GEN y[, GEN z])` yields the true Euclidean quotient of `x` and the `t_INT` or `t_POL` `y`.

`GEN gdiventsg(long s, GEN y[, GEN z])`, as `gdivent` except that `x` is a `long`.

`GEN gdiventgs[z](GEN x, long s[, GEN z])`, as `gdivent` except that `y` is a `long`.

`GEN gmod[z](GEN x, GEN y[, GEN z])` yields the remainder of `x` modulo the `t_INT` or `t_POL` `y`. A `t_REAL` or `t_FRAC` `y` is also allowed, in which case the remainder is the unique real r such that $0 \leq r < |\mathtt{y}|$ and $\mathtt{y} = q\mathtt{x} + r$ for some (in fact unique) integer q.

`GEN gmodsg(long s, GEN y[, GEN z])` as `gmod`, except `x` is a `long`.

`GEN gmodgs(GEN x, long s[, GEN z])` as `gmod`, except `y` is a `long`.

`GEN gdivmod(GEN x, GEN y, GEN *r)` If `r` is not equal to `NULL` or `ONLY_REM`, creates the (false) Euclidean quotient of `x` and `y`, and puts (the address of) the remainder into `*r`. If `r` is equal to `NULL`, do not create the remainder, and if `r` is equal to `ONLY_REM`, create and output only the remainder. The remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`.

`GEN poldivrem(GEN x, GEN y, GEN *r)` same as **gdivmod** but specifically for `t_POL`s `x` and `y`, not necessarily in the same variable. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

`GEN gdeuc(GEN x, GEN y)` creates the Euclidean quotient of the `t_POL`s `x` and `y`. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

`GEN grem(GEN x, GEN y)` creates the Euclidean remainder of the `t_POL` `x` divided by the `t_POL` `y`. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

`GEN gdivround(GEN x, GEN y)` if `x` and `y` are `t_INT`, as `diviiround`. Operate componentwise if `x` is a `t_COL`, `t_VEC` or `t_MAT`. Otherwise as **gdivent**.

`GEN centermod_i(GEN x, GEN y, GEN y2)`, as `centermodii`, componentwise.

`GEN centermod(GEN x, GEN y)`, as `centermod_i`, except that `y2` is computed (and left on the stack for efficiency).

`GEN ginvmod(GEN x, GEN y)` creates the inverse of `x` modulo `y` when it exists. `y` must be of type `t_INT` (in which case `x` is of type `t_INT`) or `t_POL` (in which case `x` is either a scalar type or a `t_POL`).

8.10 GCD, content and primitive part.

8.10.1 Generic.

`GEN resultant(GEN x, GEN y)` creates the resultant of the `t_POL`s `x` and `y` computed using Sylvester's matrix (inexact inputs), a modular algorithm (inputs in $\mathbf{Q}[X]$) or the subresultant algorithm, as optimized by Lazard and Ducos. Either of `x` and `y` may also be scalars (treated as polynomials of degree 0)

`GEN ggcd(GEN x, GEN y)` creates the GCD of `x` and `y`.

`GEN glcm(GEN x, GEN y)` creates the LCM of `x` and `y`.

`GEN gbezout(GEN x,GEN y, GEN *u,GEN *v)` returns the GCD of `x` and `y`, and puts (the addresses of) objects u and v such that $u\mathtt{x} + v\mathtt{y} = \gcd(\mathtt{x}, \mathtt{y})$ into `*u` and `*v`.

`GEN subresext(GEN x, GEN y, GEN *U, GEN *V)` returns the resultant of `x` and `y`, and puts (the addresses of) polynomials u and v such that $u\mathtt{x} + v\mathtt{y} = \mathrm{Res}(\mathtt{x}, \mathtt{y})$ into `*U` and `*V`.

`GEN content(GEN x)` returns the GCD of all the components of `x`.

`GEN primitive_part(GEN x, GEN *c)` sets `c` to `content(x)` and returns the primitive part `x / c`. A trivial content is set to `NULL`.

`GEN primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

8.10.2 Over the rationals.

`long Q_pval(GEN x, GEN p)` valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.

`long Q_pvalrem(GEN x, GEN p, GEN *r)` returns the valuation e at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`. The quotient $\mathtt{x}/\mathtt{p}^e$ is returned in `*r`.

`GEN Q_abs(GEN x)` absolute value of the `t_INT` or `t_FRAC` `x`.

`GEN Q_abs_shallow(GEN x)` x being a `t_INT` or a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(`x`)` otherwise.

`GEN Q_gcd(GEN x, GEN y)` gcd of the `t_INT` or `t_FRAC` `x` and `y`.

In the following functions, arguments belong to a $M \otimes_{\mathbf{Z}} \mathbf{Q}$ for some natural $\mathbf{Z}$-module M, e.g. multivariate polynomials with integer coefficients (or vectors/matrices recursively built from such objects), and an element of M is said to be *integral*. We are interested in contents, denominators, etc. with respect to this canonical integral structure; in particular, contents belong to $\mathbf{Q}$, denominators to $\mathbf{Z}$. For instance the $\mathbf{Q}$-content of $(1/2)xy$ is $(1/2)$, and its $\mathbf{Q}$-denominator is 2, whereas `content` would return $y/2$ and `denom` 1.

`GEN Q_content(GEN x)` the $\mathbf{Q}$-content of x

`GEN Q_denom(GEN x)` the $\mathbf{Q}$-denominator of x. Shallow function.

`GEN Q_primitive_part(GEN x, GEN *c)` sets `c` to the $\mathbf{Q}$-content of `x` and returns `x / c`, which is integral.

`GEN Q_primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN Q_remove_denom(GEN x, GEN *ptd)` sets `d` to the **Q**-denominator of `x` and returns `x * d`, which is integral. Shallow function.

`GEN Q_div_to_int(GEN x, GEN c)` returns `x / c`, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_mul_to_int(GEN x, GEN c)` returns `x * c`, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_muli_to_int(GEN x, GEN d)` returns `x * c`, assuming c is a `t_INT` and the result is integral.

`GEN mul_content(GEN cx, GEN cy)` `cx` and `cy` are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their product (either a `GEN` or `NULL`).

`GEN mul_denom(GEN dx, GEN dy)` `dx` and `dy` are as set by `Q_remove_denom`: either a `t_INT` or `NULL` representing the trivial denominator 1. Returns their product (either a `t_INT` or `NULL`).

8.11 Generic arithmetic operators.

8.11.1 Unary operators.

`GEN gneg[z](GEN x[, GEN z])` yields $-$`x`.

`GEN gneg_i(GEN x)` shallow function yielding $-$`x`.

`GEN gabs[z](GEN x[, GEN z])` yields |`x`|.

`GEN gsqr(GEN x)` creates the square of `x`.

`GEN ginv(GEN x)` creates the inverse of `x`.

8.11.2 Binary operators.

Let "*op*" be a binary operation among

op=**add**: addition (`x + y`).

op=**sub**: subtraction (`x - y`).

op=**mul**: multiplication (`x * y`).

op=**div**: division (`x / y`).

The names and prototypes of the functions corresponding to *op* are as follows:

`GEN g`*op*`(GEN x, GEN y)`

`GEN g`*op*`gs(GEN x, long s)`

`GEN g`*op*`sg(long s, GEN y)`

Explicitly

`GEN gadd(GEN x, GEN y)`, `GEN gaddgs(GEN x, long s)`, `GEN gaddsg(GEN s, GEN x)`

`GEN gmul(GEN x, GEN y)`, `GEN gmulgs(GEN x, long s)`, `GEN gmulsg(GEN s, GEN x)`

`GEN gsub(GEN x, GEN y)`, `GEN gsubgs(GEN x, long s)`, `GEN gsubsg(GEN s, GEN x)`

`GEN gdiv(GEN x, GEN y)`, `GEN gdivgs(GEN x, long s)`, `GEN gdivsg(GEN s, GEN x)`

`GEN gpow(GEN x, GEN y, long l)` creates $\texttt{x}^{\texttt{y}}$. If `y` is a `t_INT`, return `powgi(x,y)` (the precision `l` is not taken into account). Otherwise, the result is $\exp(\texttt{y} * \log(\texttt{x}))$ where exact arguments are converted to floats of precision `l` in case of need; if there is no need, for instance if x is a `t_REAL`, l is ignored. Indeed, if x is a `t_REAL`, the accuracy of $\log x$ is determined from the accuracy of x, it is no problem to multiply by y, even if it is an exact type, and the accuracy of the exponential is determined, exactly as in the case of the initial $\log x$.

`GEN gpowgs(GEN x, long n)` creates $\texttt{x}^{\texttt{n}}$ using binary powering. To treat the special case $n = 0$, we consider `gpowgs` as a series of `gmul`, so we follow the rule of returning result which is as exact as possible given the input. More precisely, we return

• `gen_1` if x has type `t_INT`, `t_REAL`, `t_FRAC`, or `t_PADIC`

• `Mod(1,N)` if x is a `t_INTMOD` modulo N.

• `gen_1` for `t_COMPLEX`, `t_QUAD` unless one component is a `t_INTMOD`, in which case we return `Mod(1, N)` for a suitable N (the gcd of the moduli that appear).

• `FF_1`(x) for a `t_FFELT`.

• `RgX_get_1`(x) for a `t_POL`.

• `qfi_1`(x) and `qfr_1`(x) for `t_QFI` and `t_QFR`.

• the identity permutation for `t_VECSMALL`.

• etc. Of course, the only practical use of this routine for $n = 0$ is to obtain the multiplicative neutral element in the base ring (or to treat marginal cases that should be special cased anyway if there is the slightest doubt about what the result should be).

`GEN powgi(GEN x, GEN y)` creates $\texttt{x}^{\texttt{y}}$, where `y` is a `t_INT`, using left-shift binary powering. The case where $y = 0$ (as all cases where y is small) is handled by `gpowgs(x, 0)`.

In addition we also have the obsolete forms:

```
void gaddz(GEN x, GEN y, GEN z)

void gsubz(GEN x, GEN y, GEN z)

void gmulz(GEN x, GEN y, GEN z)

void gdivz(GEN x, GEN y, GEN z)
```

8.12 Generic operators: product, powering, factorback.

`GEN divide_conquer_prod(GEN v, GEN (*mul)(GEN,GEN))` v is a vector of objects, which can be "multiplied" using the `mul` function. Return the "product" of the $v[i]$ using a product tree: by convention return `gen_1` if v is the empty vector, a copy of $v[1]$ if it has a single entry; and otherwise apply the function recursively on the vector (twice smaller)

$\texttt{mul}(v[1], v[2])$, $\texttt{mul}(v[3], v[4])$, ...

Only requires that `mul` is an associative binary operator, which need not correspond to a true multiplication. `D` is meant to encode an arbitrary evaluation context, set it to `NULL` in simple cases where you do not need this. Leaves some garbage on stack, but suitable for `gerepileupto` if `mul` is.

To describe the following functions, we use the following private typedefs to simplify the description:

```
typedef (*F0)(void *);
typedef (*F1)(void *, GEN);
typedef (*F2)(void *, GEN, GEN);
```

They correspond to generic functions with one and two arguments respectively (the `void*` argument provides some arbitrary evaluation context).

`GEN divide_conquer_assoc(GEN v, void *D, F2 op)` general version of `divide_conquer_prod`. Given two objects x, y, assume that `op(D, x, y)` implements an associative binary operator. If v has k entries, return

$$v[1]\ op\ v[2]\ op\ \ldots\ op\ v[k];$$

returns `gen_1` if $k = 0$ and a copy of $v[1]$ if $k = 1$.

`GEN gen_pow(GEN x, GEN n, void *D, F1 sqr, F2 mul)` $n > 0$ a `t_INT`, returns x^n; `mul(D, x, y)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D, x, x)`.

`GEN gen_powu(GEN x, ulong n, void *D, F1 sqr, F2 mul)` $n > 0$, returns x^n. See `gen_pow`.

`GEN gen_pow_i(GEN x, GEN n, void *E, F1 sqr, F2 mul)` internal variant of `gen_pow`, not memory-clean.

`GEN gen_powu_i(GEN x, ulong n, void *E, F1 sqr, F2 mul)` internal variant of `gen_powu`, not memory-clean.

`GEN gen_pow_fold(GEN x, GEN n, void *D, F1 sqr, F1 msqr)` variant of `gen_pow`, where `mul` is replaced by `msqr`, with `msqr(D, y)` returning xy^2. In particular `D` must implicitly contain x.

`GEN gen_pow_fold_i(GEN x, GEN n, void *E, F1 sqr, F1 msqr)` internal variant of the function `gen_pow_fold`, not memory-clean.

`GEN gen_powu_fold(GEN x, ulong n, void *D, F1 sqr, F1 msqr)`, see `gen_pow_fold`.

`GEN gen_powu_fold_i(GEN x, ulong n, void *E, F1 sqr, F1 msqr)` see `gen_pow_fold_i`.

`GEN gen_powers(GEN x, long n, long usesqr, void *D, F1 sqr, F2 mul, F0 one)` returns $[\texttt{x}^0, \ldots, \texttt{x}^\texttt{n}]$ as a `t_VEC`; `mul(D, x, y)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D, x, x)`; `one` returns the monoid unit. The flag `usesqr` should be set to 1 if squaring are faster than multiplication by x.

`GEN gen_factorback(GEN L, GEN e, F2 mul, F2 pow, void *D)` generic form of `factorback`. The pair $[L, e]$ is of the form

- `[fa, NULL]`, `fa` a two-column factorization matrix: expand it.
- `[v, NULL]`, v a vector of objects: return their product.
- or `[v, e]`, v a vector of objects, e a vector of integral exponents: return the product of the $v[i]^{e[i]}$.

`mul(D, x, y)` and `pow(D, x, n)` return xy and x^n respectively.

8.13 Matrix and polynomial norms.

This section concerns only standard norms of $\mathbf{R}$ and $\mathbf{C}$ vector spaces, not algebraic norms given by the determinant of some multiplication operator. We have already seen type-specific functions like `ZM_supnorm` or `RgM_fpnorml2` and limit ourselves to generic functions assuming nothing about their `GEN` argument; these functions allow the following scalar types: `t_INT`, `t_FRAC`, `t_REAL`, `t_COMPLEX`, `t_QUAD` and are defined recursively (in terms of norms of their components) for the following "container" types: `t_POL`, `t_VEC`, `t_COL` and `t_MAT`. They raise an error if some other type appears in the argument.

`GEN gnorml2(GEN x)` The norm of a scalar is the square of its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the *square* of the usual L^2 norm. In most applications, the missing square root computation can be skipped.

`GEN gnorml1(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the the usual L^1 norm. One must include a real precision `prec` in case the inputs include `t_COMPLEX` or `t_QUAD` with exact rational components: a square root must be computed and we must choose an accuracy.

`GEN gnorml1_fake(GEN x)` as `gnorml1`, except that the norm of a `t_QUAD` $x + wy$ or `t_COMPLEX` $x + Iy$ is defined as $|x| + |y|$, where we use the ordinary real absolute value. This is still a norm of $\mathbf{R}$ vector spaces, which is easier to compute than `gnorml1` and can often be used in its place.

`GEN gsupnorm(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the max of the norms of its components. A precision `prec` must be included for the same reason as in `gnorml1`.

`void gsupnorm_aux(GEN x, GEN *m, GEN *m2, long prec)` Low-level function underlying `gsupnorm`, used as follows:

```
GEN m = NULL, m2 = NULL;
gsupnorm_aux(x, &m, &m2);
```

After the call, the sup norm of x is the min of `m` and the square root of `m2`; one or both of `m`, `m2` may be `NULL`, in which case it must be omitted. You may initially set `m` and `m2` to non-`NULL` values, in which case, the above procedure yields the max of (the initial) `m`, the square root of (the initial) `m2`, and the sup norm of x.

The strange interface is due to the fact that $|z|^2$ is easier to compute than $|z|$ for a `t_QUAD` or `t_COMPLEX` z: `m2` is the max of those $|z|^2$, and `m` is the max of the other $|z|$.

8.14 Substitution and evaluation.

`GEN gsubst(GEN x, long v, GEN y)` substitutes the object `y` into `x` for the variable number `v`.

`GEN poleval(GEN q, GEN x)` evaluates the `t_POL` or `t_RFRAC` q at x. For convenience, a `t_VEC` or `t_COL` is also recognized as the `t_POL gtovecrev(q)`.

`GEN RgX_RgM_eval(GEN q, GEN x)` evaluates the `t_POL` q at the square matrix x.

`GEN RgX_RgMV_eval(GEN f, GEN V)` returns the evaluation `f(x)`, assuming that `V` was computed by `FpXQ_powers(x,`n`)` for some $n > 1$.

`GEN RgX_RgM_eval_col(GEN q, GEN x, long c)` evaluates the `t_POL` q at the square matrix x but only returns the `c`-th column of the result.

`GEN qfeval(GEN q, GEN x)` evaluates the quadratic form q (symmetric matrix) at x (column vector of compatible dimensions).

`GEN qfevalb(GEN q, GEN x, GEN y)` evaluates the polar bilinear form associated to the quadratic form q (symmetric matrix) at x, y (column vectors of compatible dimensions).

`GEN hqfeval(GEN q, GEN x)` evaluates the Hermitian form q (a Hermitian complex matrix) at x.

`GEN qf_apply_RgM(GEN q, GEN M)` q is a symmetric $n \times n$ matrix, M an $n \times k$ matrix, return $M'qM$.

`GEN qf_apply_ZM(GEN q, GEN M)` as above assuming that both q and M have integer entries.

Chapter 9:
Miscellaneous mathematical functions

9.1 Fractions.

`GEN absfrac(GEN x)` returns the absolute value of the `t_FRAC` x.

`GEN absfrac_shallow(GEN x)` x being a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(x)` otherwise.

`GEN sqrfrac(GEN x)` returns the square of the `t_FRAC` x.

9.2 Complex numbers.

`GEN imag(GEN x)` returns a copy of the imaginary part of `x`.

`GEN real(GEN x)` returns a copy of the real part of `x`. If `x` is a `t_QUAD`, returns the coefficient of 1 in the "canonical" integral basis $(1, \omega)$.

The last two functions are shallow, and not suitable for `gerepileupto`:

`GEN imag_i(GEN x)` as `gimag`, returns a pointer to the imaginary part. `GEN real_i(GEN x)` as `greal`, returns a pointer to the real part.

`GEN mulreal(GEN x, GEN)` returns the real part of xy; x, y have type `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`. See also `RgM_mulreal`.

`GEN cxnorm(GEN x)` norm of the `t_COMPLEX` x (modulus squared).

`GEN cxexpm1(GEN x)` returns $\exp(x) - 1$, for a `t_COMPLEX` x.

9.3 Quadratic numbers and binary quadratic forms.

`GEN quad_disc(GEN x)` returns the discriminant of the `t_QUAD` x.

`GEN quadnorm(GEN x)` norm of the `t_QUAD` x.

`GEN qfb_disc(GEN x)` returns the discriminant of the `t_QFI` or `t_QFR` `x`.

`GEN qfb_disc3(GEN x, GEN y, GEN z)` returns $y^2 - 4xz$ assuming all inputs are `t_INT`s. Not stack-clean.

9.4 Polynomials.

`GEN truecoeff(GEN x, long n)` returns `polcoeff0(x,n, -1)`, i.e. the coefficient of the term of degree `n` in the main variable.

`GEN polcoeff_i(GEN x, long n, long v)` internal shallow function. Rewrite x as a Laurent polynomial in the variable v and returns its coefficient of degree n (`gen_0` if this falls outside the coefficient array). Allow `t_POL`, `t_SER`, `t_RFRAC` and scalars.

`long degree(GEN x)` returns `poldegree(x, -1)`, the degree of `x` with respect to its main variable, with the usual meaning if the leading coefficient of x is non-zero. If the sign of x is 0, this function always returns -1. Otherwise, we return the index of the leading coefficient of x, i.e. the coefficient of largest index stored in x. For instance the "degrees" of

```
0. E-38 * x^4 + 0.E-19 * x + 1
Mod(0,2) * x^0    \\ sign is 0 !
```

are 4 and -1 respectively.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL x` with respect to its main variable, *if* its leading coefficient is non-zero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1.

`GEN characteristic(GEN x)` returns the characteristic of the base ring over which the polynomial is defined (as defined by `t_INTMOD` and `t_FFELT` components). The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

`GEN residual_characteristic(GEN x)` returns a kind of "residual characteristic" of the base ring over which the polynomial is defined. This is defined as the gcd of all moduli `t_INTMOD`s occurring in the structure, as well as primes p arising from `t_PADIC`s or `t_FFELT`s. The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

`GEN resultant(GEN x,GEN y)` resultant of `x` and `y`, with respect to the main variable of highest priority. Uses either the subresultant algorithm (generic case), a modular algorithm (inputs in $\mathbf{Q}[X]$) or Sylvester's matrix (inexact inputs).

`GEN resultant2(GEN x, GEN y)` resultant of `x` and `y`, with respect to the main variable of highest priority. Computes the determinant of Sylvester's matrix.

`GEN resultant_all(GEN u, GEN v, GEN *sol)` returns `resultant(x,y)`. If `sol` is not `NULL`, sets it to the last non-constant remainder in the polynomial remainder sequence if such a sequence was computed, and to `gen_0` otherwise (e.g. polynomials of degree 0, u, v in $\mathbf{Q}[X]$).

`GEN cleanroots(GEN x, long prec)` returns the complex roots of the complex polynomial x (with coefficients `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX` of the above). The roots are returned as `t_REAL` or `t_COMPLEX` of `t_REAL`s of precision `prec` (guaranteeing a non-0 imaginary part). See `QX_complex_roots`.

`GEN polmod_to_embed(GEN x, long prec)` return the vector of complex embeddings of the `t_POLMOD` x (with complex coefficients). Shallow function, simple complex variant of `conjvec`.

9.5 Power series.

`GEN derivser(GEN x)` returns the derivative of the `t_SER` `x` with respect to its main variable.

`GEN integser(GEN x)` returns the primitive of the `t_SER` `x` with respect to its main variable.

`GEN truecoeff(GEN x, long n)` returns `polcoeff0(x,n, -1)`, i.e. the coefficient of the term of degree `n` in the main variable.

`GEN ser_unscale(GEN P, GEN h)` return $P(hx)$, not memory clean.

`GEN ser_normalize(GEN x)` divide x by its "leading term" so that the series is either 0 or equal to $t^v(1+O(t))$. Shallow function if the "leading term" is 1.

9.6 Functions to handle `t_FFELT`.

These functions define the public interface of the `t_FFELT` type to use in generic functions. However, in specific functions, it is better to use the functions class `FpXQ` and/or `Flxq` as appropriate.

`GEN FF_p(GEN a)` returns the characteristic of the definition field of the `t_FFELT` element `a`.

`long FF_f(GEN a)` returns the dimension of the definition field over its prime field; the cardinality of the dimension field is thus p^f.

`GEN FF_p_i(GEN a)` shallow version of `FF_p`.

`GEN FF_q(GEN a)` returns the cardinal of the definition field of the `t_FFELT` element `a`.

`GEN FF_mod(GEN a)` returns the polynomial (with reduced `t_INT` coefficients) defining the finite field, in the variable used to display a.

`GEN FF_to_FpXQ(GEN a)` converts the `t_FFELT` `a` to a polynomial P with reduced `t_INT` coefficients such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g.

`GEN FF_to_FpXQ_i(GEN a)` shallow version of `FF_to_FpXQ`.

`GEN FF_to_F2xq(GEN a)` converts the `t_FFELT` `a` to a `F2x` P such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g. This only work if the characteristic is 2.

`GEN FF_to_F2xq_i(GEN a)` shallow version of `FF_to_F2xq`.

`GEN FF_to_Flxq(GEN a)` converts the `t_FFELT` `a` to a `Flx` P such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g. This only work if the characteristic is small enough.

`GEN FF_to_Flxq_i(GEN a)` shallow version of `FF_to_Flxq`.

`GEN p_to_FF(GEN p, long v)` returns a `t_FFELT` equal to 1 in the finite field $\mathbf{Z}/p\mathbf{Z}$. Useful for generic code that wants to handle (inefficiently) $\mathbf{Z}/p\mathbf{Z}$ as if it were not a prime field.

`GEN FF_1(GEN a)` returns the unity in the definition field of the `t_FFELT` element `a`.

`GEN FF_zero(GEN a)` returns the zero element of the definition field of the `t_FFELT` element `a`.

`int FF_equal0(GEN a)`, `int FF_equal1(GEN a)`, `int FF_equalm1(GEN a)` returns 1 if the `t_FFELT` `a` is equal to 0 (resp. 1, resp. -1) else 0.

`int FF_equal(GEN a, GEN b)` return 1 if the `t_FFELT` `a` and `b` have the same definition field and are equal, else 0.

`int FF_samefield(GEN a, GEN b)` return 1 if the `t_FFELT` `a` and `b` have the same definition field, else 0.

`int Rg_is_FF(GEN c, GEN *ff)` to be called successively on many objects, setting `*ff = NULL` (unset) initially. Returns 1 as long as c is a `t_FFELT` defined over the same field as `*ff` (setting `*ff = c` if unset), and 0 otherwise.

`int RgC_is_FFC(GEN x, GEN *ff)` apply `Rg_is_FF` successively to all components of the `t_VEC` or `t_COL` x. Return 0 if one call fails, and 1 otherwise.

`int RgM_is_FFM(GEN x, GEN *ff)` apply `Rg_is_FF` to all components of the `t_MAT`. Return 0 if one call fails, and 1 otherwise.

`GEN FF_add(GEN a, GEN b)` returns $a + b$ where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_Z_add(GEN a, GEN x)` returns $a + x$, where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

`GEN FF_Q_add(GEN a, GEN x)` returns $a + x$, where `a` is a `t_FFELT`, and `x` is a `t_RFRAC`, the computation being performed in the definition field of `a`.

`GEN FF_sub(GEN a, GEN b)` returns $a - b$ where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_mul(GEN a, GEN b)` returns ab where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_Z_mul(GEN a, GEN b)` returns ab, where `a` is a `t_FFELT`, and `b` is a `t_INT`, the computation being performed in the definition field of `a`.

`GEN FF_div(GEN a, GEN b)` returns a/b where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_neg(GEN a)` returns $-a$ where `a` is a `t_FFELT`.

`GEN FF_neg_i(GEN a)` shallow function returning $-a$ where `a` is a `t_FFELT`.

`GEN FF_inv(GEN a)` returns a^{-1} where `a` is a `t_FFELT`.

`GEN FF_sqr(GEN a)` returns a^2 where `a` is a `t_FFELT`.

`GEN FF_mul2n(GEN a, long n)` returns $a2^n$ where `a` is a `t_FFELT`.

`GEN FF_pow(GEN x, GEN n)` returns a^n where `a` is a `t_FFELT` and`n` is a `t_INT`.

`GEN FF_Z_Z_muldiv(GEN a, GEN x, GEN y)` returns ay/z, where `a` is a `t_FFELT`, and `x` and `y` are `t_INT`, the computation being performed in the definition field of `a`.

`GEN Z_FF_div(GEN x, GEN a)` return x/a where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

`GEN FF_norm(GEN a)` returns the norm of the `t_FFELT` `a` with respect to its definition field.

`GEN FF_trace(GEN a)` returns the trace of the `t_FFELT` `a` with respect to its definition field.

`GEN FF_conjvec(GEN a)` returns the vector of conjugates $[a, a^p, a^{p^2}, \ldots, a^{p^{n-1}}]$ where the `t_FFELT` `a` belong to a field with p^n elements.

`GEN FF_charpoly(GEN a)` returns the characteristic polynomial) of the `t_FFELT a` with respect to its definition field.

`GEN FF_minpoly(GEN a)` returns the minimal polynomial of the `t_FFELT a`.

`GEN FF_sqrt(GEN a)` returns an `t_FFELT` b such that $a = b^2$ if it exist, where `a` is a `t_FFELT`.

`long FF_issquareall(GEN x, GEN *pt)` returns 1 if `x` is a square, and 0 otherwise. If `x` is indeed a square, set `pt` to its square root.

`long FF_issquare(GEN x)` returns 1 if `x` is a square and 0 otherwise.

`long FF_ispower(GEN x, GEN K, GEN *pt)` Given K a positive integer, returns 1 if `x` is a K-th power, and 0 otherwise. If `x` is indeed a K-th power, set `pt` to its K-th root.

`GEN FF_sqrtn(GEN a, GEN n, GEN *zn)` returns an `n`-th root of `a` if it exist. If `zn` is non-`NULL` set it to a primitive `n`-th root of the unity.

`GEN FF_log(GEN a, GEN g, GEN o)` the `t_FFELT g` being a generator for the definition field of the `t_FFELT a`, returns a `t_INT` e such that $a^e = g$. If e does not exists, the result is currently undefined. If `o` is not `NULL` it is assumed to be a factorization of the multiplicative order of `g` (as set by `FF_primroot`)

`GEN FF_order(GEN a, GEN o)` returns the order of the `t_FFELT a`. If `o` is non-`NULL`, it is assumed that `o` is a multiple of the order of `a`.

`GEN FF_primroot(GEN a, GEN *o)` returns a generator of the multiplicative group of the definition field of the `t_FFELT a`. If `o` is not `NULL`, set it to the factorization of the order of the primitive root (to speed up `FF_log`).

`GEN FFX_factor(GEN f, GEN a)` returns the factorization of the univariate polynomial `f` over the definition field of the `t_FFELT a`. The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFX_roots(GEN f, GEN a)` returns the roots (`t_FFELT`) of the univariate polynomial `f` over the definition field of the `t_FFELT a`. The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFM_FFC_mul(GEN M, GEN C, GEN ff)` returns the product of the matrix `M` (`t_MAT`) and the column vector `C` (`t_COL`) over the finite field given by `ff` (`t_FFELT`).

`GEN FFM_ker(GEN M, GEN ff)` returns the kernel of the `t_MAT M` defined over the finite field given by the `t_FFELT ff` (obtained by `RgM_is_FFM(M,\&ff)`).

`GEN FFM_det(GEN M, GEN ff)`

`GEN FFM_image(GEN M, GEN ff)`

`GEN FFM_inv(GEN M, GEN ff)`

`GEN FFM_mul(GEN M, GEN N, GEN ff)` returns the product of the matrices `M` and `N` (`t_MAT`) over the finite field given by `ff` (`t_FFELT`).

`long FFM_rank(GEN M, GEN ff)`

9.7 Transcendental functions.

The following two functions are only useful when interacting with gp, to manipulate its internal default precision (expressed as a number of decimal digits, not in words as used everywhere else):

`long getrealprecision(void)` returns `realprecision`.

`long setrealprecision(long n, long *prec)` sets the new `realprecision` to n, which is returned. As a side effect, set `prec` to the corresponding number of words `ndec2prec(n)`.

9.7.1 Transcendental functions with t_REAL arguments.

In the following routines, x is assumed to be a `t_REAL` and the result is a `t_REAL` (sometimes a `t_COMPLEX` with `t_REAL` components), with the largest accuracy which can be deduced from the input. The naming scheme is inconsistent here, since we sometimes use the prefix `mp` even though `t_INT` inputs are forbidden:

`GEN sqrtr(GEN x)` returns the square root of x.

`GEN sqrtnr(GEN x, long n)` returns the n-th root of x, assuming $n \geq 1$ and $x > 0$. Not stack clean.

`GEN mpcos[z](GEN x[, GEN z])` returns $\cos(x)$.

`GEN mpsin[z](GEN x[, GEN z])` returns $\sin(x)$.

`GEN mplog[z](GEN x[, GEN z])` returns $\log(x)$. We must have $x > 0$ since the result must be a `t_REAL`. Use `glog` for the general case, where you want such computations as $\log(-1) = I$.

`GEN mpexp[z](GEN x[, GEN z])` returns $\exp(x)$.

`GEN mpexpm1(GEN x)` returns $\exp(x) - 1$, but is more accurate than `subrs(mpexp(x), 1)`, which suffers from catastrophic cancellation if $|x|$ is very small.

`void mpsincosm1(GEN x, GEN *s, GEN *c)` sets s and c to $\sin(x)$ and $\cos(x) - 1$ respectively, where x is a `t_REAL`; the latter is more accurate than `subrs(mpcos(y), 1)`, which suffers from catastrophic cancellation if $|x|$ is very small.

`GEN mpveceint1(GEN C, GEN eC, long n)` as `veceint1`; assumes that $C > 0$ is a `t_REAL` and that `eC` is `NULL` or `mpexp(C)`.

`GEN mpeint1(GEN x, GEN expx)` returns `eint1`(x), for a `t_REAL` $x \geq 0$, assuming that `expx` is `mpexp`(x).

`GEN szeta(long s, long prec)` returns the value of Riemann's zeta function at the (possibly negative) integer $s \neq 1$, in relative accuracy `prec`.

`GEN mplambertW(GEN y)` solution x of the implicit equation $x\exp(x) = y$, for $y > 0$ a `t_REAL`.

Useful low-level functions which *disregard* the sign of x:

`GEN sqrtr_abs(GEN x)` returns $\sqrt{|x|}$ assuming $x \neq 0$.

`GEN exp1r_abs(GEN x)` returns $\exp(|x|) - 1$, assuming $x \neq 0$.

`GEN logr_abs(GEN x)` returns $\log(|x|)$, assuming $x \neq 0$.

A few variants on sin and cos:

`void mpsincos(GEN x, GEN *s, GEN *c)` sets s and c to $\sin(x)$ and $\cos(x)$ respectively, where x is a `t_REAL`

`GEN expIr(GEN x)` returns $\exp(ix)$, where x is a `t_REAL`. The return type is `t_COMPLEX` unless the imaginary part is equal to 0 to the current accuracy (its sign is 0).

`GEN expIxy(GEN x, GEN y, long prec)` returns $\exp(ixy)$. Efficient when x is real and y pure imaginary.

`void gsincos(GEN x, GEN *s, GEN *c, long prec)` general case.

A generalization of `affrr_fixlg`

`GEN affc_fixlg(GEN x, GEN res)` assume `res` was allocated using `cgetc`, and that x is either a `t_REAL` or a `t_COMPLEX` with `t_REAL` components. Assign x to `res`, first shortening the components of `res` if needed (in a `gerepile`-safe way). Further convert `res` to a `t_REAL` if x is a `t_REAL`.

`GEN trans_eval(const char *fun, GEN (*f) (GEN, long), GEN x, long prec)` evaluate transcendental function f (named `"fun"` at the argument x and precision `prec`. This is a quick way to implement a transcendental function to be made available under GP, starting from a C function handling only `t_REAL` and `t_COMPLEX` arguments. This routine first converts x to a suitable type:

- `t_INT`/`t_FRAC` to `t_REAL` of precision `prec`, `t_QUAD` to `t_REAL` or `t_COMPLEX` of precision `prec`.
- `t_POLMOD` to a `t_COL` of complex embeddings (as in `conjvec`)

Then evaluates the function at `t_VEC`, `t_COL`, `t_MAT` arguments coefficientwise.

9.7.2 Transcendental functions with `t_PADIC` arguments.

`GEN Qp_exp(GEN x)` shortcut for `gexp(x, /*ignored*/prec)`

`GEN Qp_gamma(GEN x)` shortcut for `ggamma(x, /*ignored*/prec)`

`GEN Qp_log(GEN x)` shortcut for `glog(x, /*ignored*/prec)`

`GEN Qp_sqrt(GEN x)` shortcut for `gsqrt(x, /*ignored*/prec)` Return `NULL` if x is not a square.

`GEN Qp_sqrtn(GEN x, GEN n, GEN *z)` shortcut for `gsqrtn(x, n, z, /*ignored*/prec)`. Return `NULL` if x is not an n-th power.

9.7.3 Cached constants.

The cached constant is returned at its current precision, which may be larger than `prec`. One should always use the `mp`*xxx* variant: `mppi`, `mpeuler`, or `mplog2`.

`GEN consteuler(long prec)` precomputes Euler-Mascheroni's constant at precision `prec`.

`GEN constcatalan(long prec)` precomputes Catalan's constant at precision `prec`.

`GEN constpi(long prec)` precomputes π at precision `prec`.

`GEN constlog2(long prec)` precomputes $\log(2)$ at precision `prec`.

`void mpbern(long n, long prec)` precomputes the n even Bernoulli numbers $B_2, \ldots, B_{2n}$ as `t_FRAC` or `t_REAL`s of precision `prec`. For any $2 \leq k \leq 2n$, if a floating point approximation of B_k to accuracy `prec` is enough to reconstruct it exactly, a `t_FRAC` is stored; otherwise a `t_REAL`

at the requested accuracy. No more than n Bernoulli numbers will ever be stored (by `bernfrac` or `bernreal`), unless a subsequent call to `mpbern` increases the cache. If `prec` is 0, the B_k are computed exactly.

The following functions use cached data if `prec` is smaller than the precision of the cached value; otherwise the newly computed data replaces the old cache.

`GEN mppi(long prec)` returns π at precision `prec`.

`GEN Pi2n(long n, long prec)` returns $2^n\pi$ at precision `prec`.

`GEN PiI2(long n, long prec)` returns the complex number $2\pi i$ at precision `prec`.

`GEN PiI2n(long n, long prec)` returns the complex number $2^n\pi i$ at precision `prec`.

`GEN mpeuler(long prec)` returns Euler-Mascheroni's constant at precision `prec`.

`GEN mpeuler(long prec)` returns Catalan's number at precision `prec`.

`GEN mplog2(long prec)` returns $\log 2$ at precision `prec`.

`GEN bernreal(long i, long prec)` returns the Bernoulli number B_i as a `t_REAL` at precision `prec`. If `mpbern(n, p)` was called previously with $n \geq i$ and $p \geq$ `prec`, then the cached value is (converted to a `t_REAL` of accuracy `prec` then) returned. Otherwise, the missing value is computed. In the latter case, if $n \geq i$, the cached table is updated.

`GEN bernfrac(long i)` returns the Bernoulli number B_i as a rational number (`t_FRAC` or `t_INT`). If a cached table includes B_i as a rational number, the latter is returned. Otherwise, the missing value is computed. In the latter case, the cached Bernoulli table may be updated.

9.8 Permutations .

Permutation are represented in two different ways

- (`perm`) a `t_VECSMALL` p representing the bijection $i \mapsto p[i]$; unless mentioned otherwise, this is the form used in the functions below for both input and output,
- (`cyc`) a `t_VEC` of `t_VECSMALL`s representing a product of disjoint cycles.

`GEN identity_perm(long n)` return the identity permutation on n symbols.

`GEN cyclic_perm(long n, long d)` return the cyclic permutation mapping i to $i + d \pmod n$ in S_n. Assume that $d \leq n$.

`GEN perm_mul(GEN s, GEN t)` multiply s and t (composition $s \circ t$)

`GEN perm_conj(GEN s, GEN t)` return sts^{-1}.

`int perm_commute(GEN p, GEN q)` return 1 if p and q commute, 0 otherwise.

`GEN perm_inv(GEN p)` returns the inverse of p.

`GEN perm_pow(GEN p, long n)` returns p^n

`GEN cyc_pow_perm(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `perm`).

`GEN cyc_pow(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `cyc`).

`GEN perm_cycles(GEN p)` return the cyclic decomposition of p.

`long perm_order(GEN p)` returns the order of the permutation p (as the lcm of its cycle lengths).

`GEN vecperm_orbits(GEN p, long n)` the permutation $p \in S_n$ being given as a product of disjoint cycles, return the orbits of the subgroup generated by p on $\{1, 2, \ldots, n\}$.

9.9 Small groups.

The small (finite) groups facility is meant to deal with subgroups of Galois groups obtained by `galoisinit` and thus is currently limited to weakly super-solvable groups.

A group *grp* of order n is represented by its regular representation (for an arbitrary ordering of its element) in S_n. A subgroup of such group is represented by the restriction of the representation to the subgroup. A *small group* can be either a group or a subgroup. Thus it is embedded in some S_n, where n is the multiple of the order. Such n is called the *domain* of the small group. The domain of a trivial subgroup cannot be derived from the subgroup data, so some functions require the subgroup domain as argument.

The small group *grp* is represented by a `t_VEC` with two components:

grp[1] is a generating subset $[s_1, \ldots, s_g]$ of *grp* expressed as a vector of permutation of length n.

grp[2] contains the relative orders $[o_1, \ldots, o_g]$ of the generators *grp*[1].

See `galoisinit` for the technical details.

`GEN checkgroup(GEN gal, GEN *elts)` checks whether *gal* is a small group or a Galois group. Returns the underlying small group and set *elts* to the list of elements or to `NULL` if it is not known.

`GEN galois_group(GEN gal)` return the underlying small group of the Galois group *gal*.

`GEN cyclicgroup(GEN g, long s)` returns the cyclic group with generator g of order s.

`GEN trivialgroup(void)` returns the trivial group.

`GEN dicyclicgroup(GEN g1, GEN g2, long s1, long s2)` returns the group with generators *g1*, *g2* with respecting relative orders *s1*, *s2*.

`GEN abelian_group(GEN v)` let v be a `t_VECSMALL` seen as the SNF of a small abelian group, return its regular representation.

`long group_domain(GEN grp)` returns the `domain` of the *non-trivial* small group *grp*. Return an error if *grp* is trivial.

`GEN group_elts(GEN grp, long n)` returns the list of elements of the small group *grp* of domain n as permutations.

`GEN group_set(GEN grp, long n)` returns a *F2v* b such that $b[i]$ is set if and only if the small group *grp* of domain n contains a permutation sending 1 to i.

`GEN groupelts_set(GEN elts, long n)`, where *elts* is the list of elements of a small group of domain n, returns a *F2v* b such that $b[i]$ is set if and only if the small group contains a permutation sending 1 to i.

`long group_order(GEN grp)` returns the order of the small group *grp* (which is the product of the relative orders).

`long group_isabelian(GEN grp)` returns 1 the the small group *grp* is Abelian, else 0.

`GEN group_abelianHNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns `NULL`, else returns the HNF matrix of *grp* with respect to the generating family *grp*[1]. If *elts* is no `NULL`, it must be the list of elements of *grp*.

`GEN group_abelianSNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns `NULL`, else returns its cyclic decomposition. If *elts* is no `NULL`, it must be the list of elements of *grp*.

`long group_subgroup_isnormal(GEN G, GEN H)`, H being a subgroup of the small group G, returns 1 if H is normal in G, else 0.

`long group_isA4S4(GEN grp)` returns 1 if the small group *grp* is isomorphic to A_4, 2 if it is isomorphic to S_4 and 0 else. This is mainly to deal with the idiosyncrasy of the format.

`GEN group_leftcoset(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, the the left coset gG as a vector of permutations.

`GEN group_rightcoset(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, the the right coset Gg as a vector of permutations.

`long group_perm_normalize(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, return 1 if $gGg^-1 = G$, else 0.

`GEN group_quotient(GEN G, GEN H)`, where G is a small group and H is a subgroup of G, returns the quotient map $G \to G/H$ as an abstract data structure.

`GEN quotient_perm(GEN C, GEN g)` where C is the quotient map $G \to G/H$ for some subgroup H of G and g an element of G, return the image of g by C (i.e. the coset gH).

`GEN quotient_group(GEN C, GEN G)` where C is the quotient map $G \to G/H$ for some *normal* subgroup H of G, return the quotient group G/H as a small group.

`GEN quotient_subgroup_lift(GEN C, GEN H, GEN S)` where C is the quotient map $G \to G/H$ for some group G normalizing H and S is a subgroup of G/H, return the inverse image of S by C.

`GEN group_subgroups(GEN grp)` returns the list of subgroups of the small group *grp* as a `t_VEC`.

`GEN subgroups_tableset(GEN S, long n)` where S is a vector of subgroups of domain n, returns a table which matchs the set of elements of the subgroups against the index of the subgroups.

`long tableset_find_index(GEN tbl, GEN set)` searchs the set `set` in the table `tbl` and returns its associated index, or 0 if not found.

`GEN groupelts_abelian_group(GEN elts)` where *elts* is the list of elements of an *Abelian* small group, returns the corresponding small group.

`GEN groupelts_center(GEN elts)` where *elts* is the list of elements of a small group, returns the list of elements of the center of the group.

`GEN group_export(GEN grp, long format)` exports a small group to another format, see `galoisexport`.

`long group_ident(GEN grp, GEN elts)` returns the index of the small group *grp* in the GAP4 Small Group library, see `galoisidentify`. If *elts* is not `NULL`, it must be the list of elements of *grp*.

`long group_ident_trans(GEN grp, GEN elts)` returns the index of the regular representation of the small group *grp* in the GAP4 Transitive Group library, see `polgalois`. If *elts* is no `NULL`, it must be the list of elements of *grp*.

Chapter 10:
Standard data structures

10.1 Character strings.

10.1.1 Functions returning a `char *`.

`char* pari_strdup(const char *s)` returns a malloc'ed copy of s (uses `pari_malloc`).

`char* pari_strndup(const char *s, long n)` returns a malloc'ed copy of at most n chars from s (uses `pari_malloc`). If s is longer than n, only n characters are copied and a terminal null byte is added.

`char* stack_strdup(const char *s)` returns a copy of s, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_strcat(const char *s, const char *t)` returns the concatenation of s and t, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_sprintf(const char *fmt, ...)` runs `pari_sprintf` on the given arguments, returning a string allocated on the PARI stack.

`char* itostr(GEN x)` writes the `t_INT` x to a `stack_malloc`'ed string.

`char* GENtostr(GEN x)`, using the current default output format (`GP_DATA->fmt`, which contains the output style and the number of significant digits to print), converts x to a malloc'ed string. Simple variant of `pari_sprintf`.

`char* GENtostr_unquoted(GEN x)` as `GENtostr` with the following differences: 1) a `t_STR` x is printed without enclosing quotes (to be used by `print`); 2) the result is allocated on the stack and *must not* be freed.

`char* GENtoTeXstr(GEN x)`, as `GENtostr`, except that `f_TEX` overrides the output format from `GP_DATA->fmt`.

`char* RgV_to_str(GEN g, long flag)` g being a vector of `GEN`s, returns a malloc'ed string, the concatenation of the `GENtostr` applied to its elements, except that `t_STR` are printed without enclosing quotes. `flag` determines the output format: `f_RAW`, `f_PRETTYMAT` or `f_TEX`.

10.1.2 Functions returning a t_STR.

`GEN strtoGENstr(const char *s)` returns a `t_STR` with content s.

`GEN strntoGENstr(const char *s, long n)` returns a `t_STR` containing the first n characters of s.

`GEN chartoGENstr(char c)` returns a `t_STR` containing the character c.

`GEN GENtoGENstr(GEN x)` returns a `t_STR` containing the printed form of x (in `raw` format). This is often easier to use that `GENtostr` (which returns a malloc-ed `char*`) since there is no need to free the string after use.

`GEN GENtoGENstr_nospace(GEN x)` as `GENtoGENstr`, removing all spaces from the output.

`GEN Str(GEN g)` as `RgV_to_str` with output format `f_RAW`, but returns a `t_STR`, not a malloc'ed string.

`GEN Strtex(GEN g)` as `RgV_to_str` with output format `f_TEX`, but returns a `t_STR`, not a malloc'ed string.

`GEN Strexpand(GEN g)` as `RgV_to_str` with output format `f_RAW`, performing tilde and environment expansion on the result. Returns a `t_STR`, not a malloc'ed string.

`GEN gsprintf(const char *fmt, ...)` equivalent to `pari_sprintf(fmt,...`, followed by `strtoGENstr`. Returns a `t_STR`, not a malloc'ed string.

`GEN gvsprintf(const char *fmt, va_list ap)` variadic version of `gsprintf`

10.2 Output.

10.2.1 Output contexts.

An output coutext, of type `PariOUT`, is a `struct` that models a stream and contains the following function pointers:

```
void (*putch)(char);           /* fputc()-alike */
void (*puts)(const char*);     /* fputs()-alike */
void (*flush)(void);           /* fflush()-alike */
```

The methods `putch` and `puts` are used to print a character or a string respectively. The method `flush` is called to finalize a messages.

The generic functions `pari_putc`, `pari_puts`, `pari_flush` and `pari_printf` print according to a *default output context*, which should be sufficient for most purposes. Lower level functions are available, which take an explicit output context as first argument:

`void out_putc(PariOUT *out, char c)` essentially equivalent to `out->putc(c)`. In addition, registers whether the last character printed was a `\n`.

`void out_puts(PariOUT *out, const char *s)` essentially equivalent to `out->puts(s)`. In addition, registers whether the last character printed was a `\n`.

`void out_printf(PariOUT *out, const char *fmt, ...)`

`void out_vprintf(PariOUT *out, const char *fmt, va_list ap)`

N.B. The function `out_flush` does not exist since it would be identical to `out->flush()`

`int pari_last_was_newline(void)` returns a non-zero value if the last character printed via `out_putc` or `out_puts` was `\n`, and 0 otherwise.

`void pari_set_last_newline(int last)` sets the boolean value to be returned by the function `pari_last_was_newline` to *last*.

10.2.2 Default output context. They are defined by the global variables `pariOut` and `pariErr` for normal outputs and warnings/errors, and you probably do not want to change them. If you *do* change them, diverting output in non-trivial ways, this probably means that you are rewriting `gp`. For completeness, we document in this section what the default output contexts do.

pariOut. writes output to the `FILE*` `pari_outfile`, initialized to `stdout`. The low-level methods are actually the standard `putc` / `fputs`, plus some magic to handle a log file if one is open.

pariErr. prints to the `FILE*` `pari_errfile`, initialized to `stderr`. The low-level methods are as above.

You can stick with the default `pariOut` output context and change PARI's standard output, redirecting `pari_outfile` to another file, using

`void switchout(const char *name)` where `name` is a character string giving the name of the file you want to write to; the output is *appended* at the end of the file. To close the file and revert to outputting to `stdout`, call `switchout(NULL)`.

10.2.3 PARI colors. In this section we describe the low-level functions used to implement GP's color scheme, associated to the `colors` default. The following symbolic names are associated to gp's output strings:

- `c_ERR` an error message
- `c_HIST` a history number (as in `%1 = ...`)
- `c_PROMPT` a prompt
- `c_INPUT` an input line (minus the prompt part)
- `c_OUTPUT` an output
- `c_HELP` a help message
- `c_TIME` a timer
- `c_NONE` everything else

If the `colors` default is set to a non-empty value, before gp outputs a string, it first outputs an ANSI colors escape sequence — understood by most terminals —, according to the `colors` specifications. As long as this is in effect, the following strings are rendered in color, possibly in bold or underlined.

`void term_color(long c)` prints (as if using `pari_puts`) the ANSI color escape sequence associated to output object `c`. If `c` is `c_NONE`, revert to default printing style.

`void out_term_color(PariOUT *out, long c)` as `term_color`, using output context `out`.

`char* term_get_color(char *s, long c)` returns as a character string the ANSI color escape sequence associated to output object `c`. If `c` is `c_NONE`, the value used to revert to default printing style is returned. The argument `s` is either `NULL` (string allocated on the PARI stack), or preallocated storage (in which case, it must be able to hold at least 16 chars, including the final `\0`).

10.2.4 Obsolete output functions.

These variants of `void output(GEN x)`, which prints `x`, followed by a newline and a buffer flush are complicated to use and less flexible than what we saw above, or than the `pari_printf` variants. They are provided for backward compatibility and are scheduled to disappear.

`void brute(GEN x, char format, long dec)`

`void matbrute(GEN x, char format, long dec)`

`void texe(GEN x, char format, long dec)`

10.3 Files.

The following routines are trivial wrappers around system functions (possibly around one of several functions depending on availability). They are usually integrated within PARI's diagnostics system, printing messages if `DEBUGFILES` is high enough.

`int pari_is_dir(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int pari_is_file(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int file_is_binary(FILE *f)` returns 1 if the file f is a binary file (in the `writebin` sense), 0 otherwise.

`void pari_unlink(const char *s)` deletes the file named s. Warn if the operation fails.

`void pari_fread_chars(void *b, size_t n, FILE *f)` read n chars from stream f, storing the result in pre-allocated buffer b (assumed to be large enough).

`char* path_expand(const char *s)` perform tilde and environment expansion on s. Returns a `malloc`'ed buffer.

`void strftime_expand(const char *s, char *buf, long max)` perform time expansion on s, storing the result (at most `max` chars) in buffer `buf`. Trivial wrapper around

```
time_t t = time(NULL);
strftime(but, max, s, localtime(&t);
```

`char* pari_get_homedir(const char *user)` expands `~user` constructs, returning the home directory of user `user`, or `NULL` if it could not be determined (in particular if the operating system has no such concept). The return value may point to static area and may be overwritten by subsequent system calls: use immediately or `strdup` it.

`int pari_stdin_isatty(void)` returns 1 if our standard input `stdin` is attached to a terminal. Trivial wrapper around `isatty`.

10.3.1 pariFILE.

PARI maintains a linked list of open files, to reclaim resources (file descriptors) on error or interrupts. The corresponding data structure is a `pariFILE`, which is a wrapper around a standard `FILE*`, containing further the file name, its type (regular file, pipe, input or output file, etc.). The following functions create and manipulate this structure; they are integrated within PARI's diagnostics system, printing messages if `DEBUGFILES` is high enough.

`pariFILE* pari_fopen(const char *s, const char *mode)` wrapper around `fopen(s, mode)`, return `NULL` on failure.

`pariFILE* pari_fopen_or_fail(const char *s, const char *mode)` simple wrapper around `fopen(s, mode)`; error on failure.

`pariFILE* pari_fopengz(const char *s)` opens the file whose name is s, and associates a (read-only) `pariFILE` with it. If s is a compressed file (`.gz` suffix), it is uncompressed on the fly. If s cannot be opened, also try to open s`.gz`. Returns `NULL` on failure.

`void pari_fclose(pariFILE *f)` closes the underlying file descriptor and deletes the `pariFILE` struct.

`pariFILE* pari_safefopen(const char *s, const char *mode)` creates a *new* file s (a priori for writing) with `600` permissions. Error if the file already exists. To avoid symlink attacks, a symbolic link exists, regardless of where it points to.

10.3.2 Temporary files.

PARI has its own idea of the system temp directory derived from from an environment variable (`$GPTMPDIR`, else `$TMPDIR`), or the first writable directory among `/tmp`, `/var/tmp` and `..`

`char* pari_unique_dir(const char *s)` creates a "unique directory" and return its name built from the string s, the user id and process pid (on Unix systems). This directory is itself located in the temp directory mentioned above. The name returned is `malloc`'ed.

`char* pari_unique_filename(const char *s)` creates a *new* empty file in the temp directory, whose name contains the id-string s (truncated to its first 8 chars), followed by a system-dependent suffix (incorporating the ids of both the user and the running process, for instance). The function returns the tempfile name. The name returned is `malloc`'ed.

10.4 Errors.

This section documents the various error classes, and the corresponding arguments to `pari_err`. The general syntax is

`void pari_err(numerr,...)`

In the sequel, we mostly use sequences of arguments of the form

```
const char *s
const char *fmt, ...
```

where `fmt` is a PARI format, producing a string s from the remaining arguments. Since providing the correct arguments to `pari_err` is quite error-prone, we also provide specialized routines `pari_err_`*ERRORCLASS*`(...)` instead of `pari_err(e_`*ERRORCLASS*`, ...)` so that the C compiler can check their arguments.

We now inspect the list of valid keywords (error classes) for `numerr`, and the corresponding required arguments.

10.4.1 Internal errors, "system" errors.

10.4.1.1 e_ARCH. A requested feature *s* is not available on this architecture or operating system.

```
pari_err(e_ARCH)
```

prints the error message: `sorry, 's' not available on this system`.

10.4.1.2 e_BUG. A bug in the PARI library, in function *s*.

```
pari_err(e_BUG, const char *s)
pari_err_BUG(const char *s)
```

prints the error message: `Bug in` *s*`, please report`.

10.4.1.3 e_FILE. Error while trying to open a file.

```
pari_err(e_FILE, const char *what, const char *name)
pari_err_FILE(const char *what, const char *name)
```

prints the error message: `error opening` *what*`:` '*name*'.

10.4.1.4 e_IMPL. A requested feature *s* is not implemented.

```
pari_err(e_IMPL, const char *s)
pari_err_IMPL(const char *s)
```

prints the error message: `sorry,` *s* `is not yet implemented`.

10.4.1.5 e_PACKAGE. Missing optional package *s*.

```
pari_err(e_PACKAGE, const char *s)
pari_err_PACKAGE(const char *s)
```

prints the error message: `package` *s* `is required, please install it`

10.4.2 Syntax errors, type errors.

10.4.2.1 e_DIM. arguments submitted to function *s* have inconsistent dimensions. E.g., when solving a linear system, or trying to compute the determinant of a non-square matrix.

```
pari_err(e_DIM, const char *s)
pari_err_DIM(const char *s)
```

prints the error message: `inconsistent dimensions in` *s*.

10.4.2.2 e_FLAG. A flag argument is out of bounds in function *s*.

```
pari_err(e_FLAG, const char *s)
pari_err_FLAG(const char *s)
```

prints the error message: `invalid flag in` *s*.

10.4.2.3 e_NOTFUNC. Generated by the PARI evaluator; tried to use a `GEN` which is not a `t_CLOSURE` in a function call syntax (as in `f = 1; f(2);`).

```
pari_err(e_NOTFUNC, GEN fun)
```

prints the error message: `not a function in a function call`.

10.4.2.4 e_OP. Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in `O(2) + O(3)`: it is valid to add two `t_PADIC`s, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation.

```
pari_err(e_OP, const char *op, GEN x, GEN y)
pari_err_OP(const char *op, GEN x, GEN y)
```

As `e_TYPE2`, replacing `forbidden` by `inconsistent`.

10.4.2.5 e_PRIORITY. object o in function s contains variables whose priority is incompatible with the expected operation. E.g. `Pol([x,1], 'y)`: this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable.

```
pari_err(e_PRIORITY, const char *s, GEN o, const char *op, long v)
pari_err_PRIORITY(const char *s, GEN o, const char *op, long v)
```

prints the error message: `incorrect priority in` s`, variable` v_o op v, were v_o is `gvar(o)`.

10.4.2.6 e_SYNTAX. Syntax error, generated by the PARI parser.

```
pari_err(e_SYNTAX, const char *msg, const char *e, const char *entry)
```

where `msg` is a complete error message, and `e` and `entry` point into the *same* character string, which is the input that was incorrectly parsed: `e` points to the character where the parser failed, and `entry` $\leq$ `e` points somewhat before.

Prints the error message: `msg`, followed by a colon, then a part of the input character string (in general `entry` itself, but an initial segment may be truncated if `e` $-$ `entry` is large); a caret points at `e`, indicating where the error took place.

10.4.2.7 e_TYPE. An argument x of function s had an unexpected type. (As in `factor("blah")`.)

```
pari_err(e_TYPE, const char *s, GEN x)
pari_err_TYPE(const char *s, GEN x)
```

prints the error message: `incorrect type in` s `(t_`x`)`, where `t_`x is the type of x.

10.4.2.8 e_TYPE2. Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in `Mod(1,2) + Pi`.)

```
pari_err(e_TYPE2, const char *op, GEN x, GEN y)
pari_err_TYPE2(const char *op, GEN x, GEN y)
```

prints the error message: `forbidden` s `t_`x op `t_`y, where `t_`z denotes the type of z. Here, s denotes the spelled out name of the operator $op \in \{+, *, /, \%, =\}$, e.g. *addition* for `"+"` or *assignment* for `"="`. If op is not in the above operator, list, it is taken to be the already spelled out name of a function, e.g. `"gcd"`, and the error message becomes `forbidden` op `t_`x, `t_`y.

10.4.2.9 e_VAR. polynomials x and y submitted to function s have inconsistent variables. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfinit(x^2+1)`.

```
pari_err(e_VAR, const char *s, GEN x, GEN y)
pari_err_VAR(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent variables in` s X `!=` Y, where X and Y are the names of the variables of x and y, respectively.

10.4.3 Overflows.

10.4.3.1 e_COMPONENT. Trying to access an inexistent component in a vector/matrix/list in a function: the index is less than 1 or greater than the allowed length.

```
pari_err(e_COMPONENT, const char *f, const char *op, GEN lim, GEN x)
pari_err_COMPONENT(const char *f, const char *op, GEN lim, GEN x)
```

prints the error message: `non-existent component in` *f*`:  index` *op lim*. Special case: if *f* is the empty string (no meaningful public function name can be used), we ignore it and print the message: `non-existent component:  index` *op lim*.

10.4.3.2 e_DOMAIN. An argument *x* is not in the function's domain (as in `moebius(0)` or `zeta(1)`).

```
pari_err(e_DOMAIN, char *f, char *v, char *op, GEN lim, GEN x)
pari_err_DOMAIN(char *f, char *v, char *op, GEN lim, GEN x)
```

prints the error message: `domain error in` *f*`:  v op lim`. Special case: if *op* is the empty string, we ignore *lim* and print the error message: `domain error in` *f*`:  v out of range`.

10.4.3.3 e_MAXPRIME. A function using the precomputed list of prime numbers ran out of primes.

```
pari_err(e_MAXPRIME, ulong c)
pari_err_MAXPRIME(ulong c)
```

prints the error message: `not enough precomputed primes, need primelimit ~`*c* if *c* is non-zero. And simply `not enough precomputed primes` otherwise.

10.4.3.4 e_MEM. A call to `pari_malloc` or `pari_realloc` failed.

```
pari_err(e_MEM)
```

prints the error message: `not enough memory`.

10.4.3.5 e_OVERFLOW. An object in function *s* becomes too large to be represented within PARI's hardcoded limits. (As in `2^2^2^10` or `exp(1e100)`, which overflow in `lg` and `expo`.)

```
pari_err(e_OVERFLOW, const char *s)
pari_err_OVERFLOW(const char *s)
```

prints the error message: `overflow in` *s*.

10.4.3.6 e_PREC. Function *s* fails because input accuracy is too low. (As in `floor(1e100)` at default accuracy.)

```
pari_err(e_PREC, const char *s)
pari_err_PREC(const char *s)
```

prints the error message: `precision too low in` *s*.

10.4.3.7 e_STACK. The PARI stack overflows.

```
pari_err(e_STACK)
```

prints the error message: `the PARI stack overflows !` as well as some statistics concerning stack usage.

10.4.4 Errors triggered intentionally.

10.4.4.1 e_ALARM. A timeout, generated by the `alarm` function.

```
pari_err(e_ALARM, const char *fmt, ...)
```

prints the error message: s.

10.4.4.2 e_USER. A user error, as triggered by `error`$(g_1, \ldots, g_n)$ in GP.

```
pari_err(e_USER, GEN g)
```

prints the error message: `user error:`, then the entries of the vector g.

10.4.5 Mathematical errors.

10.4.5.1 e_CONSTPOL. An argument of function s is a constant polynomial, which does not make sense. (As in `galoisinit(Pol(1))`.)

```
pari_err(e_CONSTPOL, const char *s)
pari_err_CONSTPOL(const char *s)
```

prints the error message: `constant polynomial in` s.

10.4.5.2 e_COPRIME. Function s expected two coprime arguments, and did receive x, y which were not.

```
pari_err(e_COPRIME, const char *s, GEN x, GEN y)
pari_err_COPRIME(const char *s, GEN x, GEN y)
```

prints the error message: `elements not coprime in` s`:` x, y.

10.4.5.3 e_INV. Tried to invert a non-invertible object x.

```
pari_err(e_INV, GEN x)
pari_err_INV(GEN x)
```

prints the error message: `impossible inverse:` x. If $x =$ `Mod`(a, b) is a `t_INTMOD` and a is not 0 mod b, this allows to factor the modulus, as `gcd`(a, b) is a non-trivial divisor of b.

10.4.5.4 e_IRREDPOL. Function s expected an irreducible polynomial, and did not receive one. (As in `nfinit(x^2-1)`.)

```
pari_err(e_IRREDPOL, const char *s, GEN x)
pari_err_IRREDPOL(const char *s, GEN x)
```

prints the error message: `not an irreducible polynomial in` s`:` x.

10.4.5.5 e_MISC. Generic uncategorized error.

```
pari_err(e_MISC, const char *fmt, ...)
```

prints the error message: s.

10.4.5.6 e_MODULUS. moduli x and y submitted to function s are inconsistent. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfinit(t^3-2)`.

```
pari_err(e_MODULUS, const char *s, GEN x, GEN y)
pari_err_MODULUS(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent moduli in` s, then the moduli.

10.4.5.7 e_PRIME. Function s expected a prime number, and did receive p, which was not. (As in `idealprimedec(nf, 4)`.)

```
pari_err(e_PRIME, const char *s, GEN x)
pari_err_PRIME(const char *s, GEN x)
```

prints the error message: `not a prime in` s`:` x.

10.4.5.8 e_ROOTS0. An argument of function s is a zero polynomial, and we need to consider its roots. (As in `polroots(0)`.)

```
pari_err(e_ROOTS0, const char *s)
pari_err_ROOTS0(const char *s)
```

prints the error message: `zero polynomial in` s.

10.4.5.9 e_SQRTN. Tried to compute an n-th root of x, which does not exist, in function s. (As in `sqrt(Mod(-1,3))`.)

```
pari_err(e_SQRTN, GEN x)
pari_err_SQRTN(GEN x)
```

prints the error message: `not an n-th power residue in` s`:` x.

10.4.6 Miscellaneous functions.

`long name_numerr(const char *s)` return the error number corresponding to an error name. E.g. `name_numerr("e_DIM")` returns `e_DIM`.

`const char* numerr_name(long errnum)` returns the error name corresponding to an error number. E.g. `name_numerr(e_DIM)` returns `"e_DIM"`.

`char* pari_err2str(GEN err)` returns the error message that would be printed on `t_ERROR err`. The name is allocated on the PARI stack and must not be freed.

10.5 Hashtables.

A `hashtable`, or associative array, is a set of pairs (k, v) of keys and values. PARI implements general extensible hashtables for fast data retrieval: when creating a table, we may either choose to use the PARI stack, or `malloc` so as to be stack-independent. A hashtable is implemented as a table of linked lists, each list containing all entries sharing the same hash value. The table length is a prime number, which roughly doubles as the table overflows by gaining new entries; both the current number of entries and the threshold before the table grows are stored in the table. Finally the table remembers the functions used to hash the entries's keys and to test for equality two entries hashed to the same value.

An entry, or `hashentry`, contains

• a key/value pair (k, v), both of type `void*` for maximal flexibility,

• the hash value of the key, for the table hash function. This hash is mapped to a table index (by reduction modulo the table length), but it contains more information, and is used to bypass costly general equality tests if possible,

• a link pointer to the next entry sharing the same table cell.

```
typedef struct {
  void *key, *val;
  ulong hash; /* hash(key) */
  struct hashentry *next;
} hashentry;

typedef struct {
  ulong len; /* table length */
  hashentry **table; /* the table */
  ulong nb, maxnb; /* number of entries stored and max nb before enlarging */
  ulong pindex; /* prime index */
  ulong (*hash) (void *k); /* hash function */
  int (*eq) (void *k1, void *k2); /* equality test */
  int use_stack; /* use the PARI stack, resp. malloc */
} hashtable;
```

```
hashtable* hash_create(size, hash, eq, use_stack)
      ulong size;
      ulong (*hash)(void*);
      int (*eq)(void*,void*);
      int use_stack;
```

creates a hashtable with enough room to contain `size` entries. The functions `hash` and `eq` compute the hash value of keys and test keys for equality, respectively. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_insert(hashtable *h, void *k, void *v)` inserts (k, v) in hashtable h. No copy is made: k and v themselves are stored. The implementation does not prevent one to insert two entries with equal keys k, but which of the two is affected by later commands is undefined.

`hashentry* hash_search(hashtable *h, void *k)` look for an entry with key k in h. Return it if it one exists, and `NULL` if not.

`hashentry* hash_remove(hashtable *h, void *k)` deletes an entry (k, v) with key k from h and return it. (Return `NULL` if none was found.) Only the linking structures are freed, memory associated to k and v is not reclaimed.

`void hash_destroy(hashtable *h)` deletes the hashtable, by removing all entries.

Some interesting hash functions are available:

`ulong hash_str(const char *s)`

`ulong hash_str2(const char *s)` is the historical PARI string hashing function and seems to be generally inferior to `hash_str`.

`ulong hash_GEN(GEN x)`

10.6 Dynamic arrays.

A **dynamic array** is a generic way to manage stacks of data that need to grow dynamically. It allocates memory using `pari_malloc`, and is independent of the PARI stack; it even works before the `pari_init` call.

10.6.1 Initialization.

To create a stack of objects of type `foo`, we proceed as follows:

```
foo *t_foo;
pari_stack s_foo;
pari_stack_init(&s_foo, sizeof(*t_foo), (void**)t_foo);
```

Think of `s_foo` as the controlling interface, and `t_foo` as the (dynamic) array tied to it. The value of `t_foo` may be changed as you add more elements.

10.6.2 Adding elements. The following function pushes an element on the stack.

```
/* access globals t_foo and s_foo */
void push_foo(foo x)
{
  long n = pari_stack_new(&s_foo);
  t_foo[n] = x;
}
```

10.6.3 Accessing elements.

Elements are accessed naturally through the `t_foo` pointer. For example this function swaps two elements:

```
void swapfoo(long a, long b)
{
  foo x;
  if (a > s_foo.n || b > s_foo.n) pari_err_BUG("swapfoo");
  x        = t_foo[a];
  t_foo[a] = t_foo[b];
  t_foo[b] = x;
}
```

10.6.4 Stack of stacks. Changing the address of `t_foo` is not supported in general. In particular `realloc()`'ed array of stacks and stack of stacks are not supported.

10.6.5 Public interface. Let `s` be a `pari_stack` and `data` the data linked to it. The following public fields are defined:

- `s.alloc` is the number of elements allocated for `data`.
- `s.n` is the number of elements in the stack and `data[s.n-1]` is the topmost element of the stack. `s.n` can be changed as long as $0 \leq$ `s.n` $\leq$ `s.alloc` holds.

`void pari_stack_init(pari_stack *s, size_t size, void **data)` links `*s` to the data pointer `*data`, where `size` is the size of data element. The pointer `*data` is set to `NULL`, `s->n` and `s->alloc` are set to 0: the array is empty.

`void pari_stack_alloc(pari_stack *s, long nb)` makes room for `nb` more elements, i.e. makes sure that `s.alloc` $\geq$ `s.n` $+$ `nb`, possibly reallocating `data`.

`long pari_stack_new(pari_stack *s)` increases `s.n` by one unit, possibly reallocating `data`, and returns `s.n` -1.

Caveat. The following construction is incorrect because `stack_new` can change the value of `t_foo`:

```
t_foo[ pari_stack_new(&s_foo) ] = x;
```

`void pari_stack_delete(pari_stack *s)` frees `data` and resets the stack to the state immediately following `stack_init` (`s->n` and `s->alloc` are set to 0).

`void * pari_stack_pushp(pari_stack *s, void *u)` This function assumes that `*data` is of pointer type. Pushes the element `u` on the stack `s`.

`void ** pari_stack_base(pari_stack *s)` returns the address of `data`, typecast to a `void **`.

10.7 Vectors and Matrices.

10.7.1 Access and extract. See Section 8.3.1 and Section 8.3.2 for various useful constructors. Coefficients are accessed and set using `gel`, `gcoeff`, see Section 5.2.7. There are many internal functions to extract or manipulate subvectors or submatrices but, like the accessors above, none of them are suitable for `gerepileupto`. Worse, there are no type verification, nor bound checking, so use at your own risk.

`GEN shallowcopy(GEN x)` returns a `GEN` whose components are the components of x (no copy is made). The result may now be used to compute in place without destroying x. This is essentially equivalent to

```
GEN y = cgetg(lg(x), typ(x));
for (i = 1; i < lg(x); i++) y[i] = x[i];
return y;
```

except that `t_MAT` is treated specially since shallow copies of all columns are made. The function also works for non-recursive types, but is useless in that case since it makes a deep copy. If x is known to be a `t_MAT`, you may call `RgM_shallowcopy` directly; if x is known not to be a `t_MAT`, you may call `leafcopy` directly.

`GEN RgM_shallowcopy(GEN x)` returns `shallowcopy(x)`, where x is a `t_MAT`.

`GEN shallowtrans(GEN x)` returns the transpose of x, *without* copying its components, i. e., it returns a `GEN` whose components are (physically) the components of x. This is the internal function underlying `gtrans`.

`GEN shallowconcat(GEN x, GEN y)` concatenate x and y, *without* copying components, i. e., it returns a `GEN` whose components are (physically) the components of x and y.

`GEN shallowconcat1(GEN x)` x must be `t_VEC` or `t_LIST`, concatenate its elements from left to right. Shallow version of `concat1`.

`GEN shallowmatconcat(GEN v)` shallow version of `matconcat`.

`GEN shallowextract(GEN x, GEN y)` extract components of the vector or matrix x according to the selection parameter y. This is the shallow analog of `extract0(x, y, NULL)`, see `vecextract`.

`GEN RgM_minor(GEN A, long i, long j)` given a square `t_MAT` A, return the matrix with i-th row and j-th column removed.

`GEN vconcat(GEN A, GEN B)` concatenate vertically the two `t_MAT` A and B of compatible dimensions. A `NULL` pointer is accepted for an empty matrix. See `shallowconcat`.

`GEN row(GEN A, long i)` return $A[i,]$, the i-th row of the `t_MAT` A.

`GEN row_i(GEN A, long i, long j1, long j2)` return part of the i-th row of `t_MAT` A: $A[i, j_1]$, $A[i, j_1 + 1] \dots, A[i, j_2]$. Assume $j_1 \leq j_2$.

`GEN rowcopy(GEN A, long i)` return the row $A[i,]$ of the `t_MAT` A. This function is memory clean and suitable for `gerepileupto`. See `row` for the shallow equivalent.

`GEN rowslice(GEN A, long i1, long i2)` return the `t_MAT` formed by the i_1-th through i_2-th rows of `t_MAT` A. Assume $i_1 \leq i_2$.

`GEN rowpermute(GEN A, GEN p)`, p being a `t_VECSMALL` representing a list $[p_1, \dots, p_n]$ of rows of `t_MAT` A, returns the matrix whose rows are $A[p_1,], \dots, A[p_n,]$.

`GEN rowslicepermute(GEN A, GEN p, long x1, long x2)`, short for

```
rowslice(rowpermute(A,p), x1, x2)
```

(more efficient).

`GEN vecslice(GEN A, long j1, long j2)`, return $A[j_1], \dots, A[j_2]$. If A is a `t_MAT`, these correspond to *columns* of A. The object returned has the same type as A (`t_VEC`, `t_COL` or `t_MAT`). Assume $j_1 \leq j_2$.

`GEN vecsplice(GEN A, long j)` return A with j-th entry removed (`t_VEC`, `t_COL`) or j-th column removed (`t_MAT`).

`GEN vecreverse(GEN A)`. Returns a `GEN` which has the same type as A (`t_VEC`, `t_COL` or `t_MAT`), and whose components are the $A[n], \dots, A[1]$. If A is a `t_MAT`, these are the *columns* of A.

`GEN vecpermute(GEN A, GEN p)` p is a `t_VECSMALL` representing a list $[p_1, \dots, p_n]$ of indices. Returns a `GEN` which has the same type as A (`t_VEC`, `t_COL` or `t_MAT`), and whose components are $A[p_1], \dots, A[p_n]$. If A is a `t_MAT`, these are the *columns* of A.

`GEN vecslicepermute(GEN A, GEN p, long y1, long y2)` short for

```
vecslice(vecpermute(A,p), y1, y2)
```

(more efficient).

10.7.2 Componentwise operations.

The following convenience routines automate trivial loops of the form

```
for (i = 1; i < lg(a); i++) gel(v,i) = f(gel(a,i), gel(b,i))
```

for suitable f:

`GEN vecinv(GEN a)`. Given a vector a, returns the vector whose i-th component is `ginv`$(a[i])$.

`GEN vecmul(GEN a, GEN b)`. Given a and b two vectors of the same length, returns the vector whose i-th component is `gmul`$(a[i], b[i])$.

`GEN vecdiv(GEN a, GEN b)`. Given a and b two vectors of the same length, returns the vector whose i-th component is `gdiv`$(a[i], b[i])$.

`GEN vecpow(GEN a, GEN n)`. Given n a `t_INT`, returns the vector whose i-th component is $a[i]^n$.

`GEN vecmodii(GEN a, GEN b)`. Assuming a and b are two `ZV` of the same length, returns the vector whose i-th component is `modii`$(a[i], b[i])$.

Note that `vecadd` or `vecsub` do not exist since `gadd` and `gsub` have the expected behavior. On the other hand, `ginv` does not accept vector types, hence `vecinv`.

10.7.3 Low-level vectors and columns functions.

These functions handle `t_VEC` as an abstract container type of `GEN`s. No specific meaning is attached to the content. They accept both `t_VEC` and `t_COL` as input, but `col` functions always return `t_COL` and `vec` functions always return `t_VEC`.

Note. All the functions below are shallow.

`GEN const_col(long n, GEN x)` returns a `t_COL` of `n` components equal to `x`.

`GEN const_vec(long n, GEN x)` returns a `t_VEC` of `n` components equal to `x`.

`int vec_isconst(GEN v)` Returns 1 if all the components of `v` are equal, else returns 0.

`void vec_setconst(GEN v, GEN x)` v a pre-existing vector. Set all its components to x.

`int vec_is1to1(GEN v)` Returns 1 if the components of `v` are pair-wise distinct, i.e. if $i \mapsto v[i]$ is a 1-to-1 mapping, else returns 0.

`GEN vec_shorten(GEN v, long n)` shortens the vector `v` to `n` components.

`GEN vec_lengthen(GEN v, long n)` lengthens the vector `v` to `n` components. The extra components are not initialized.

`GEN vec_insert(GEN v, long n, GEN x)` inserts x at position n in the vector v.

10.8 Vectors of small integers.

10.8.1 t_VECSMALL.

These functions handle t_VECSMALL as an abstract container type of small signed integers. No specific meaning is attached to the content.

GEN const_vecsmall(long n, long c) returns a t_VECSMALL of n components equal to c.

GEN vec_to_vecsmall(GEN z) identical to ZV_to_zv(z).

GEN vecsmall_to_vec(GEN z) identical to zv_to_ZV(z).

GEN vecsmall_to_col(GEN z) identical to zv_to_ZC(z).

GEN vecsmall_copy(GEN x) makes a copy of x on the stack.

GEN vecsmall_shorten(GEN v, long n) shortens the t_VECSMALL v to n components.

GEN vecsmall_lengthen(GEN v, long n) lengthens the t_VECSMALL v to n components. The extra components are not initialized.

GEN vecsmall_indexsort(GEN x) performs an indirect sort of the components of the t_VECSMALL x and return a permutation stored in a t_VECSMALL.

void vecsmall_sort(GEN v) sorts the t_VECSMALL v in place.

long vecsmall_max(GEN v) returns the maximum of the elements of t_VECSMALL v, assumed non-empty.

long vecsmall_indexmax(GEN v) returns the index of the largest element of t_VECSMALL v, assumed non-empty.

long vecsmall_min(GEN v) returns the minimum of the elements of t_VECSMALL v, assumed non-empty.

long vecsmall_indexmin(GEN v) returns the index of the smallest element of t_VECSMALL v, assumed non-empty.

long vecsmall_isin(GEN v, long x) returns the first index i such that v[i] is equal to x. Naive search in linear time, does not assume that v is sorted.

GEN vecsmall_uniq(GEN v) given a t_VECSMALL v, return the vector of unique occurrences.

GEN vecsmall_uniq_sorted(GEN v) same as vecsmall_uniq, but assumes v sorted.

long vecsmall_duplicate(GEN v) given a t_VECSMALL v, return 0 if there is no duplicates, or the index of the first duplicate (vecsmall_duplicate([1,1]) returns 2).

long vecsmall_duplicate_sorted(GEN v) same as vecsmall_duplicate, but assume v sorted.

int vecsmall_lexcmp(GEN x, GEN y) compares two t_VECSMALL lexically.

int vecsmall_prefixcmp(GEN x, GEN y) truncate the longest t_VECSMALL to the length of the shortest and compares them lexicographically.

GEN vecsmall_prepend(GEN V, long s) prepend s to the t_VECSMALL V.

GEN vecsmall_append(GEN V, long s) append s to the t_VECSMALL V.

GEN vecsmall_concat(GEN u, GEN v) concat the t_VECSMALL u and v.

`long vecsmall_coincidence(GEN u, GEN v)` returns the numbers of indices where `u` and `v` agree.

`long vecsmall_pack(GEN v, long base, long mod)` handles the `t_VECSMALL` `v` as the digit of a number in base `base` and return this number modulo `mod`. This can be used as an hash function.

10.8.2 Vectors of `t_VECSMALL`. These functions manipulate vectors of `t_VECSMALL` (vecvecsmall).

`GEN vecvecsmall_sort(GEN x)` sorts lexicographically the components of the vector `x`.

`GEN vecvecsmall_sort_uniq(GEN x)` sorts lexicographically the components of the vector `x`, removing duplicates entries.

`GEN vecvecsmall_indexsort(GEN x)` performs an indirect lexicographic sorting of the components of the vector `x` and return a permutation stored in a `t_VECSMALL`.

`long vecvecsmall_search(GEN x, GEN y, long flag)` `x` being a sorted vecvecsmall and `y` a `t_VECSMALL`, search `y` inside `x`. `flag` has the same meaning as for `setsearch`.

Chapter 11:
Functions related to the GP interpreter

11.1 Handling closures.

11.1.1 Functions to evaluate t_CLOSURE.

`void closure_disassemble(GEN C)` print the `t_CLOSURE` `C` in GP assembly format.

`GEN closure_callgenall(GEN C, long n, ...)` evaluate the `t_CLOSURE` `C` with the `n` arguments (of type `GEN`) following `n` in the function call. Assumes `C` has arity $\geq$ `n`.

`GEN closure_callgenvec(GEN C, GEN args)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`. Assumes `C` has arity $\geq$ `lg(args)` $-$ `1`.

`GEN closure_callgen1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x`. Assumes `C` has arity ≥ 1.

`GEN closure_callgen2(GEN C, GEN x, GEN y)` evaluate the `t_CLOSURE` `C` with argument `x`, `y`. Assumes `C` has arity ≥ 2.

`void closure_callvoid1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x` and discard the result. Assumes `C` has arity ≥ 1.

The following technical functions are used to evaluate *inline* closures and closures of arity 0.

The control flow statements (break, next and return) will cause the evaluation of the closure to be interrupted; this is called below a *flow change*. When that occurs, the functions below generally return `NULL`. The caller can then adopt three positions:

- raises an exception (`closure_evalnobrk`).
- passes through (by returning NULL itself).
- handles the flow change.

`GEN closure_evalgen(GEN code)` evaluates a closure and returns the result, or `NULL` if a flow change occurred.

`GEN closure_evalnobrk(GEN code)` as `closure_evalgen` but raise an exception if a flow change occurs. Meant for iterators where interrupting the closure is meaningless, e.g. `intnum` or `sumnum`.

`void closure_evalvoid(GEN code)` evaluates a closure whose return value is ignored. The caller has to deal with eventual flow changes by calling `loop_break`.

The remaining functions below are for exceptional situations:

`GEN closure_evalres(GEN code)` evaluates a closure and returns the result. The difference with `closure_evalgen` being that, if the flow end by a `return` statement, the result will be the returned value instead of `NULL`. Used by the main GP loop.

`GEN closure_evalbrk(GEN code, long *status)` as `closure_evalres` but set `status` to a non-zero value if a flow change occurred. This variant is not stack clean. Used by the break loop.

`GEN closure_trapgen(long numerr, GEN code)` evaluates closure, while trapping error `numerr`. Return `(GEN)1L` if error trapped, and the result otherwise, or `NULL` if a flow change occurred. Used by trap.

11.1.2 Functions to handle control flow changes.

`long loop_break(void)` processes an eventual flow changes inside an iterator. If this function return 1, the iterator should stop.

11.1.3 Functions to deal with lexical local variables.

Function using the prototype code `'V'` need to manually create and delete a lexical variable for each code `'V'`, which will be given a number $-1, -2, \ldots$.

`void push_lex(GEN a, GEN code)` creates a new lexical variable whose initial value is a on the top of the stack. This variable get the number -1, and the number of the other variables is decreased by one unit. When the first variable of a closure is created, the argument `code` must be the closure that references this lexical variable. The argument `code` must be `NULL` for all subsequent variables (if any). (The closure contains the debugging data for the variable).

`void pop_lex(long n)` deletes the n topmost lexical variables, increasing the number of other variables by n. The argument n must match the number of variables allocated through `push_lex`.

`GEN get_lex(long vn)` get the value of the variable with number `vn`.

`void set_lex(long vn, GEN x)` set the value of the variable with number `vn`.

11.1.4 Functions returning new closures.

`GEN compile_str(const char *s)` returns the closure corresponding to the GP expression s.

`GEN closure_deriv(GEN code)` returns a closure corresponding to the numerical derivative of the closure `code`.

`GEN snm_closure(entree *ep, GEN data)` Let `data` be a vector of length m, `ep` be an `entree` pointing to a C function f of arity $n+m$, returns a `t_CLOSURE` object g of arity n such that $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, gel(data, 1), ..., gel(data, m))$. If `data` is `NULL`, then $m = 0$ is assumed. This function has a low overhead since it does not copy `data`.

`GEN strtofunction(char *str)` returns a closure corresponding to the built-in or install'ed function named `str`.

`GEN strtoclosure(char *str, long n, ...)` returns a closure corresponding to the built-in or install'ed function named `str` with the n last parameters set to the n `GEN`s following n, see `snm_closure`. This function has an higher overhead since it copies the parameters and does more input validation.

In the example code below, `agm1` is set to the function `x->agm(x,1)` and `res` is set to `agm(2,1)`.

```
GEN agm1 = strtoclosure("agm",1, gen_1);
GEN res = closure_callgen1(agm1, gen_2);
```

11.1.5 Functions used by the gp debugger (break loop). `long closure_context(long s)` restores the compilation context starting at frame `s+1`, and returns the index of the topmost frame. This allow to compile expressions in the topmost lexical scope.

`void closure_err(void)` prints a backtrace of the last 20 stack frames.

11.1.6 Standard wrappers for iterators. Two families of standard wrappers are provided to interface iterators like `intnum` or `sumnum` with GP.

11.1.6.1 Standard wrappers for inline closures. Theses wrappers are used to implement GP functions taking inline closures as input. The object `(GEN)E` must be an inline closure which is evaluated with the lexical variable number -1 set to x.

`GEN gp_eval(void *E, GEN x)` is used for the prototype code `'E'`.

`long gp_evalvoid(void *E, GEN x)` is used for the prototype code `'I'`. The resulting value is discarded. Return a non-zero value if a control-flow instruction request the iterator to terminate immediately.

`long gp_evalbool(void *E, GEN x)` returns the boolean `gp_eval(E, x)` evaluates to (i.e. true iff the value is non-zero).

`GEN gp_evalupto(void *E, GEN x)` memory-safe version of `gp_eval`, `gcopy`-ing the result, when the evaluator returns components of previously allocated objects (e.g. member functions).

11.1.6.2 Standard wrappers for true closures. These wrappers are used to implement GP functions taking true closures as input.

`GEN gp_call(void *E, GEN x)` evaluates the closure `(GEN)E` on x.

`long gp_callbool(void *E, GEN x)` evaluates the closure `(GEN)E` on x, returns `1` if its result is non-zero, and `0` otherwise.

`long gp_callvoid(void *E, GEN x)` evaluates the closure `(GEN)E` on x, discarding the result. Return a non-zero value if a control-flow instruction request the iterator to terminate immediately.

11.2 Defaults.

`int pari_is_default(const char *s)` return 1 if s is the name of a default, 0 otherwise.

`GEN setdefault(const char *s, const char *v, long flag)` is the low-level function underlying `default0`. If s is `NULL`, call all default setting functions with string argument `NULL` and flag `d_ACKNOWLEDGE`. Otherwise, check whether s corresponds to a default and call the corresponding default setting function with arguments v and *flag*.

We shall describe these functions below: if v is `NULL`, we only look at the default value (and possibly print or return it, depending on `flag`); otherwise the value of the default to v, possibly after some translation work. The flag is one of

- `d_INITRC` called while reading the `gprc`: print and return `gnil`, possibly defer until `gp` actually starts.
- `d_RETURN` return the current value, as a `t_INT` if possible, as a `t_STR` otherwise.
- `d_ACKNOWLEDGE` print the current value, return `gnil`.

- `d_SILENT` print nothing, return `gnil`.

Low-level functions called by `setdefault`:

```
GEN sd_TeXstyle(const char *v, long flag)
GEN sd_colors(const char *v, long flag)
GEN sd_compatible(const char *v, long flag)
GEN sd_datadir(const char *v, long flag)
GEN sd_debug(const char *v, long flag)
GEN sd_debugfiles(const char *v, long flag)
GEN sd_debugmem(const char *v, long flag)
GEN sd_factor_add_primes(const char *v, long flag)
GEN sd_factor_proven(const char *v, long flag)
GEN sd_format(const char *v, long flag)
GEN sd_histsize(const char *v, long flag)
GEN sd_log(const char *v, long flag)
GEN sd_logfile(const char *v, long flag)
GEN sd_nbthreads(const char *v, long flag)
GEN sd_new_galois_format(const char *v, long flag)
GEN sd_output(const char *v, long flag)
GEN sd_parisize(const char *v, long flag)
GEN sd_path(const char *v, long flag)
GEN sd_prettyprinter(const char *v, long flag)
GEN sd_primelimit(const char *v, long flag)
GEN sd_realprecision(const char *v, long flag)
GEN sd_recover(const char *v, long flag)
GEN sd_secure(const char *v, long flag)
GEN sd_seriesprecision(const char *v, long flag)
GEN sd_simplify(const char *v, long flag)
GEN sd_sopath(char *v, int flag)
GEN sd_strictargs(const char *v, long flag)
GEN sd_strictmatch(const char *v, long flag)
GEN sd_threadsize(const char *v, long flag)
```

Generic functions used to implement defaults: most of the above routines are implemented in terms of the following generic ones. In all routines below

• `v` and `flag` are the arguments passed to `default`: `v` is a new value (or the empty string: no change), and `flag` is one of `d_INITRC`, `d_RETURN`, etc.

• `s` is the name of the default being changed, used to display error messages or acknowledgements.

`GEN sd_toggle(const char *v, long flag, const char *s, int *ptn)`

• if `v` is neither `"0"` nor `"1"`, an error is raised using `pari_err`.

• `ptn` points to the current numerical value of the toggle (1 or 0), and is set to the new value (when `v` is non-empty).

For instance, here is how the timer default is implemented internally:

```
GEN
sd_timer(const char *v, long flag)
{ return sd_toggle(v,flag,"timer", &(GP_DATA->chrono)); }
```

The exact behavior and return value depends on `flag`:

• `d_RETURN`: returns the new toggle value, as a `GEN`.

• `d_ACKNOWLEDGE`: prints a message indicating the new toggle value and return `gnil`.

• other cases: print nothing and return `gnil`.

`GEN sd_ulong(const char *v, long flag, const char *s, ulong *ptn, ulong Min, ulong Max, const char **msg)`

• `ptn` points to the current numerical value of the toggle, and is set to the new value (when `v` is non-empty).

• `Min` and `Max` point to the minimum and maximum values allowed for the default.

• `v` must translate to an integer in the allowed ranger, a suffix among `k`/`K` ($\times 10^3$), `m`/`M` ($\times 10^6$), or `g`/`G` ($\times 10^9$) is allowed, but no arithmetic expression.

• `msg` is a [NULL]-terminated array of messages or `NULL` (ignored). If `msg` is not `NULL`, `msg`[i] contains a message associated to the value i of the default. The last entry in the `msg` array is used as a message associated to all subsequent ones.

The exact behavior and return value depends on `flag`:

• `d_RETURN`: returns the new toggle value, as a `GEN`.

• `d_ACKNOWLEDGE`: prints a message indicating the new value, possibly a message associated to it via the `msg` argument, and return `gnil`.

• other cases: print nothing and return `gnil`.

`GEN sd_string(const char *v, long flag, const char *s, char **pstr)` • `v` is subjet to environment expansion, then time expansion.

• `pstr` points to the current string value, and is set to the new value (when `v` is non-empty).

11.3 Records and Lazy vectors.

The functions in this section are used to implement `ell` structures and analogous objects, which are vectors some of whose components are initialized to dummy values, later computed on demand. We start by initializing the structure:

`GEN obj_init(long d, long n)` returns an *obj* S, a `t_VEC` with d regular components, accessed as `gel(S,1)`, ..., `gel(S,d)`; together with a record of n members, all initialized to 0. The arguments d and n must be non-negative.

After `S = obj_init(d, n)`, the prototype of our other functions are of the form

```
GEN obj_do(GEN S, long tag, ...)
```

The first argument S holds the structure to be managed. The second argument *tag* is the index of the struct member (from 1 to n) we operate on. We recommend to define an `enum` and use descriptive names instead of hardcoded numbers. For instance, if $n = 3$, after defining

```
enum { TAG_p = 1, TAG_list, TAG_data };
```

one may use `TAG_list` or 2 indifferently as a tag. The former being preferred, of course.

Technical note. In the current implementation, S is a `t_VEC` with $d + 1$ entries. The first d components are ordinary `t_GEN` entries, which you can read or assign to in the customary way. But the last component `gel(S,d+1)`, a `t_VEC` of length n initialized to `zerovec`(n), must be handled in a special way: you should never access or modify its components directly, only through the API we are about to describe. Indeed, its entries are meant to contain dynamic data, which will be stored, retrieved and replaced (for instance by a value computed to a higher accuracy), while interacting safely with intermediate `gerepile` calls. This mechanism allows to simulate C `struct`s, in a simpler way than with general hashtables, while remaining compatible with the GP language, which knows neither structs nor hashtables. It also serialize the structure in an ordinary `GEN`, which facilitates copies and garbage collection (use `gcopy` or `gerepile`), rather than having to deal with individual components of actual C `struct`s.

`GEN obj_check(GEN S, long tag)` if the *tag*-component in S is non empty, return it. Otherwise return `NULL`. The `t_INT` 0 (initial value) is used as a sentinel to indicated an empty component.

`GEN obj_insert(GEN S, long tag, GEN O)` insert (a clone of) O as *tag*-component of S. Any previous value is deleted, and data pointing to it become invalid.

`GEN obj_insert_shallow(GEN S, long K, GEN O)` as `obj_insert`, inserting O as-is, not via a clone.

`GEN obj_checkbuild(GEN S, long tag, GEN (*build)(GEN))` if the *tag*-component of S is non empty, return it. Otherwise insert (a clone of) `build(S)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_padicprec(GEN S, long tag, GEN (*build)(GEN,long), long prec)` if the *tag*-component of S is non empty *and* has relative p-adic precision $\geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_prec(GEN S, long tag, GEN (*build)(GEN,long), long prec)` if the *tag*-component of S is non empty *and* has `gprecision` $\geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`void obj_free(GEN S)` destroys all clones stored in the n tagged components, and replace them by the initial value 0. The regular entries of S are unaffected, and S remains a valid object. This is used to avoid memory leaks.

Chapter 12:
Technical Reference Guide for Algebraic Number Theory

12.1 General Number Fields.

12.1.1 Number field types.

None of the following routines thoroughly check their input: they distinguish between *bona fide* structures as output by PARI routines, but designing perverse data will easily fool them. To give an example, a square matrix will be interpreted as an ideal even though the **Z**-module generated by its columns may not be an $\mathbf{Z}_K$-module (i.e. the expensive `nfisideal` routine will *not* be called).

`long nftyp(GEN x)`. Returns the type of number field structure stored in `x`, `typ_NF`, `typ_BNF`, or `typ_BNR`. Other answers are possible, meaning `x` is not a number field structure.

`GEN get_nf(GEN x, long *t)`. Extract an *nf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_bnf(GEN x, long *t)`. Extract a `bnf` structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_nfpol(GEN x, GEN *nf)` try to extract an *nf* structure from `x`, and sets `*nf` to `NULL` (failure) or to the *nf*. Returns the (monic, integral) polynomial defining the field.

`GEN get_bnfpol(GEN x, GEN *bnf, GEN *nf)` try to extract a *bnf* and an *nf* structure from `x`, and sets `*bnf` and `*nf` to `NULL` (failure) or to the corresponding structure. Returns the (monic, integral) polynomial defining the field.

`GEN checknf(GEN x)` if an *nf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_nf` is often more flexible.

`GEN checkbnf(GEN x)` if an *bnf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_bnf` is often more flexible.

`void checkbnr(GEN bnr)` Raise an exception if the argument is not a *bnr* structure.

`void checkbnrgen(GEN bnr)` Raise an exception if the argument is not a *bnr* structure, complete with explicit generators for the ray class group. This is normally useless and `checkbnr` should be instead, unless you are absolutely certain that the generators will be needed at a later point, and you are about to embark in a costly intermediate computation. PARI functions do check that generators are present in *bnr* before accessing them: they will raise an error themselves; many functions that may require them, e.g. `bnrconductor`, often do not actually need them.

`void checkrnf(GEN rnf)` Raise an exception if the argument is not an *rnf* structure.

`void checkbid(GEN bid)` Raise an exception if the argument is not a *bid* structure.

`GEN checkgal(GEN x)` if a *galoisinit* structure can be extracted from `x`, return it; otherwise raise an exception.

`void checksqmat(GEN x, long N)` check whether `x` is a square matrix of dimension `N`. May be used to check for ideals if `N` is the field degree.

`void checkprid(GEN pr)` Raise an exception if the argument is not a prime ideal structure.

`GEN get_prid(GEN ideal)` return the underlying prime ideal structure if one can be extracted from `ideal` (ideal or extended ideal), and return `NULL` otherwise.

`void checkabgrp(GEN v)` Raise an exception if the argument is not an abelian group structure, i.e. a `t_VEC` with either 2 or 3 entries: $[N, cyc]$ or $[N, cyc, gen]$.

`GEN abgrp_get_no(GEN x)` extract the cardinality N from an abelian group structure.

`GEN abgrp_get_cyc(GEN x)` extract the elementary divisors *cyc* from an abelian group structure.

`GEN abgrp_get_gen(GEN x)` extract the generators *gen* from an abelian group structure.

`void checkmodpr(GEN modpr)` Raise an exception if the argument is not a prime ideal structure.

`GEN checknfelt_mod(GEN nf, GEN x, const char *s)` given an *nf* structure `nf` and a `t_POLMOD` `x`, return the associated polynomial representative (shallow) if `x` and `nf` are compatible. Raise an exception otherwise. Set s to the name of the caller for a meaningful error message.

`void check_ZKmodule(GEN x, const char *s)` check whether x looks like $\mathbf{Z}_K$-module (a pair $[A, I]$, where A is a matrix and I is a list of ideals; A has as many columns as I has elements. Otherwise raises an exception. Set s to the name of the caller for a meaningful error message.

`long idealtyp(GEN *ideal, GEN *fa)` The input is `ideal`, a pointer to an ideal (or extended ideal), which is usually modified, `fa` being set as a side-effect. Returns the type of the underlying ideal among `id_PRINCIPAL` (a number field element), `id_PRIME` (a prime ideal) `id_MAT` (an ideal in matrix form).

If `ideal` pointed to an ideal, set `fa` to `NULL`, and possibly simplify `ideal` (for instance the zero ideal is replaced by `gen_0`). If it pointed to an extended ideal, replace `ideal` by the underlying ideal and set `fa` to the factorization matrix component.

12.1.2 Extracting info from a `nf` structure.

These functions expect a true *nf* argument associated to a number field $K = \mathbf{Q}[x]/(T)$, e.g. a *bnf* will not work. Let $n = [K : \mathbf{Q}]$ be the field degree.

`GEN nf_get_pol(GEN nf)` returns the polynomial T (monic, in $\mathbf{Z}[x]$).

`long nf_get_varn(GEN nf)` returns the variable number of the number field defining polynomial.

`long nf_get_r1(GEN nf)` returns the number of real places r_1.

`long nf_get_r2(GEN nf)` returns the number of complex places r_2.

`void nf_get_sign(GEN nf, long *r1, long *r2)` sets r_1 and r_2 to the number of real and complex places respectively. Note that $r_1 + 2r_2$ is the field degree.

`long nf_get_degree(GEN nf)` returns the number field degree, $n = r_1 + 2r_2$.

`GEN nf_get_disc(GEN nf)` returns the field discriminant.

`GEN nf_get_index(GEN nf)` returns the index of T, i.e. the index of the order generated by the power basis $(1, x, \ldots, x^{n-1})$ in the maximal order of K.

`GEN nf_get_zk(GEN nf)` returns a basis $(w_1, w_2, \dots, w_n)$ for the maximal order of K. Those are polynomials in $\mathbf{Q}[x]$ of degree $< n$; it is guaranteed that $w_1 = 1$.

`GEN nf_get_invzk(GEN nf)` returns the matrix $(m_{i,j}) \in M_n(\mathbf{Z})$ giving the power basis (x^i) in terms of the (w_j), i.e such that $x^{j-1} = \sum_{i=1}^n m_{i,j} w_i$ for all $1 \le j \le n$; since $w_1 = 1 = x^0$, we have $m_{i,1} = \delta_{i,1}$ for all i. The conversion functions in the `algtobasis` family essentially amount to a left multiplication by this matrix.

`GEN nf_get_roots(GEN nf)` returns the r_1 real roots of the polynomial defining the number fields: first the r_1 real roots (as `t_REAL`s), then the r_2 representatives of the pairs of complex conjugates.

`GEN nf_get_allroots(GEN nf)` returns all the complex roots of T: first the r_1 real roots (as `t_REAL`s), then the r_2 pairs of complex conjugates.

`GEN nf_get_M(GEN nf)` returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K: $M[i, j]$ contains $w_j(\alpha_i)$, where α_i is the i-th element of `nf_get_roots(nf)`. In particular, if v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i] w_i$ of K, then `RgM_RgC_mul(M,v)` represents the embeddings of v.

`GEN nf_get_G(GEN nf)` returns a $n \times n$ real matrix G such that $Gv \cdot Gv = T_2(v)$, where v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i] w_i$ of K and T_2 is the standard Euclidean form on $K \otimes \mathbf{R}$, i.e. $T_2(v) = \sum_\sigma |\sigma(v)|^2$, where σ runs through all n complex embeddings of K.

`GEN nf_get_roundG(GEN nf)` returns a rescaled version of G, rounded to nearest integers, specifically `RM_round_maxrank`(G).

`GEN nf_get_Tr(GEN nf)` returns the matrix of the Trace quadratic form on the basis $(w_1, \dots, w_n)$: its (i, j) entry is $\mathrm{Tr} w_i w_j$.

`GEN nf_get_diff(GEN nf)` returns the primitive part of the inverse of the above Trace matrix.

`long nf_get_prec(GEN nf)` returns the precision (in words) to which the *nf* was computed.

12.1.3 Extracting info from a `bnf` structure.

These functions expect a true *bnf* argument, e.g. a *bnr* will not work.

`GEN bnf_get_nf(GEN bnf)` returns the underlying *nf*.

`GEN bnf_get_clgp(GEN bnf)` returns the class group in *bnf*, which is a 3-component vector $[h, cyc, gen]$.

`GEN bnf_get_cyc(GEN bnf)` returns the elementary divisors of the class group (cyclic components) $[d_1, \dots, d_k]$, where $d_k \mid \dots \mid d_1$.

`GEN bnf_get_gen(GEN bnf)` returns the generators $[g_1, \dots, g_k]$ of the class group. Each g_i has order d_i, and the full module of relations between the g_i is generated by the $d_i g_i = 0$.

`GEN bnf_get_no(GEN bnf)` returns the class number.

`GEN bnf_get_reg(GEN bnf)` returns the regulator.

`GEN bnf_get_logfu(GEN bnf)` returns (complex floating point approximations to) the logarithms of the complex embeddings of our system of fundamental units.

`GEN bnf_get_fu(GEN bnf)` returns the fundamental units. Raise an error if the *bnf* does not contain units in algebraic form.

`GEN bnf_get_fu_nocheck(GEN bnf)` as `bnf_get_fu` without checking whether units are present. Do not use this unless you initialize the *bnf* yourself!

`GEN bnf_get_tuU(GEN bnf)` returns a generator of the torsion part of $\mathbf{Z}_K^*$.

`long bnf_get_tuN(GEN bnf)` returns the order of the torsion part of $\mathbf{Z}_K^*$, i.e. the number of roots of unity in K.

12.1.4 Extracting info from a `bnr` structure.

These functions expect a true *bnr* argument.

`GEN bnr_get_bnf(GEN bnr)` returns the underlying *bnf*.

`GEN bnr_get_nf(GEN bnr)` returns the underlying *nf*.

`GEN bnr_get_clgp(GEN bnr)` returns the ray class group.

`GEN bnr_get_no(GEN bnr)` returns the ray class number.

`GEN bnr_get_cyc(GEN bnr)` returns the elementary divisors of the ray class group (cyclic components) $[d_1, \dots, d_k]$, where $d_k \mid \dots \mid d_1$.

`GEN bnr_get_gen(GEN bnr)` returns the generators $[g_1, \dots, g_k]$ of the ray class group. Each g_i has order d_i, and the full module of relations between the g_i is generated by the $d_i g_i = 0$. Raise a generic error if the *bnr* does not contain the ray class group generators.

`GEN bnr_get_gen_nocheck(GEN bnr)` as `bnr_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bnr* yourself!

`GEN bnr_get_bid(GEN bnr)` returns the *bid* associated to the *bnr* modulus.

`GEN bnr_get_mod(GEN bnr)` returns the modulus associated to the *bnr*.

12.1.5 Extracting info from an `rnf` structure.

These functions expect a true *rnf* argument, associated to an extension L/K, $K = \mathbf{Q}[y]/(T)$, $L = K[x]/(P)$.

`long rnf_get_degree(GEN rnf)` returns the *relative* degree $[L : K]$.

`long rnf_get_absdegree(GEN rnf)` returns the absolute degree $[L : \mathbf{Q}]$.

`long rnf_get_nfdegree(GEN rnf)` returns the degree of the base field $[K : \mathbf{Q}]$.

`GEN rnf_get_nf(GEN rnf)` returns the base field K, an *nf* structure.

`GEN rnf_get_nfpol(GEN rnf)` returns the polynomial T defining the base field K.

`long rnf_get_nfvarn(GEN rnf)` returns the variable y associated to the base field K.

`void rnf_get_nfzk(GEN rnf, GEN *b, GEN *cb)` returns the integer basis of the base field K.

`GEN rnf_get_pol(GEN rnf)` returns the relative polynomial defining L/K.

`long rnf_get_varn(GEN rnf)` returns the variable x associated to L.

`GEN rnf_get_zk(GEN nf)` returns the relative integer basis generating $\mathbf{Z}_L$ as a $\mathbf{Z}_K$-module, as a pseudo-matrix (A, I) in HNF.

`GEN rnf_get_disc(GEN rnf)` is the output $[\mathfrak{d}, s]$ of `rnfdisc`

`GEN rnf_get_index(GEN rnf)` is the index ideal $\mathfrak{f}$

`GEN rnf_get_polabs(GEN rnf)` returns an absolute polynomial defining $L/\mathbf{Q}$.

`GEN rnf_get_invzk(GEN rnf)` contains A^{-1}, where (A, I) is the chosen pseudo-basis for $\mathbf{Z}_L$ over $\mathbf{Z}_K$.

`GEN rnf_get_map(GEN rnf)` returns technical data associated to the map $K \to L$. Currently, this contains data from `rnfequation`, as well as the polynomials T and P.

12.1.6 Extracting info from a `bid` structure.

These functions expect a true *bid* argument, associated to a modulus I in a number field K.

`GEN bid_get_mod(GEN bid)` returns the modulus associated to the *bid*.

`GEN bid_get_grp(GEN bid)` returns the Abelian group associated to $(\mathbf{Z}_K/I)^*$modulus associated to the *bid*.

`GEN bid_get_ideal(GEN bid)` return the finite part of the *bid* modulus (an integer ideal).

`GEN bid_get_arch(GEN bid)` return the Archimedean part of the *bid* modulus (a vector of real places).

`GEN bid_get_no(GEN bid)` returns the cardinality of the group $(\mathbf{Z}_K/I)^*$.

`GEN bid_get_cyc(GEN bid)` returns the elementary divisors of the group $(\mathbf{Z}_K/I)^*$ (cyclic components) $[d_1, \ldots, d_k]$, where $d_k \mid \ldots \mid d_1$.

`GEN bid_get_gen(GEN bid)` returns the generators of $(\mathbf{Z}_K/I)^*$ contained in *bid*. Raise a generic error if *bid* does not contain generators.

`GEN bid_get_gen_nocheck(GEN bid)` as `bid_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bid* yourself!

12.1.7 Increasing accuracy.

`GEN nfnewprec(GEN x, long prec)`. Raise an exception if `x` is not a number field structure (*nf*, *bnf* or *bnr*). Otherwise, sets its accuracy to `prec` and return the new structure. This is mostly useful with `prec` larger than the accuracy to which `x` was computed, but it is also possible to decrease the accuracy of `x` (truncating relevant components, which may speed up later computations). This routine may modify the original `x` (see below).

This routine is straightforward for *nf* structures, but for the other ones, it requires all principal ideals corresponding to the *bnf* relations in algebraic form (they are originally only available via floating point approximations). This in turn requires many calls to `bnfisprincipal0`, which is often slow, and may fail if the initial accuracy was too low. In this case, the routine will not actually fail but recomputes a *bnf* from scratch!

Since this process may be very expensive, the corresponding data is cached (as a *clone*) in the *original* `x` so that later precision increases become very fast. In particular, the copy returned by `nfnewprec` also contains this additional data.

`GEN bnfnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnf* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN bnrnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnr* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN nfnewprec_shallow(GEN nf, long prec)`

`GEN bnfnewprec_shallow(GEN bnf, long prec)`

`GEN bnrnewprec_shallow(GEN bnr, long prec)` Shallow functions underlying the above, except that the first argument must now have the corresponding number field type. I.e. one cannot call `nfnewprec_shallow(nf, prec)` if `nf` is actually a *bnf*.

12.1.8 Number field arithmetic. The number field $K = \mathbf{Q}[X]/(T)$ is represented by an `nf` (or `bnf` or `bnr` structure). An algebraic number belonging to K is given as

- a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo T), or
- a `t_POLMOD` (modulo T), or
- a `t_COL v` of dimension $N = [K : \mathbf{Q}]$, representing the element in terms of the computed integral basis (e_i), as

```
sum(i = 1, N, v[i] * nf.zk[i])
```

The preferred forms are `t_INT` and `t_COL` of `t_INT`. Routines can handle denominators but it is much more efficient to remove denominators first (`Q_remove_denom`) and take them into account at the end.

Safe routines. The following routines do not assume that their `nf` argument is a true *nf* (it can be any number field type, e.g. a *bnf*), and accept number field elements in all the above forms. They return their result in `t_COL` form.

`GEN nfadd(GEN nf, GEN x, GEN y)` returns $x + y$.

`GEN nfdiv(GEN nf, GEN x, GEN y)` returns x/y.

`GEN nfinv(GEN nf, GEN x)` returns x^{-1}.

`GEN nfmul(GEN nf,GEN x,GEN y)` returns xy.

`GEN nfpow(GEN nf,GEN x,GEN k)` returns x^k, k is in $\mathbf{Z}$.

`GEN nfpow_u(GEN nf,GEN x, ulong k)` returns x^k, $k \geq 0$.

`GEN nfsqr(GEN nf,GEN x)` returns x^2.

`long nfval(GEN nf, GEN x, GEN pr)` returns the valuation of x at the maximal ideal `p` associated to the *prid* `pr`. Returns `LONG_MAX` is x is 0.

`GEN nfnorm(GEN nf,GEN x)` absolute norm of x.

`GEN nftrace(GEN nf,GEN x)` absolute trace of x.

`GEN nfpoleval(GEN nf, GEN pol, GEN a)` evaluate the `t_POL pol` (with coefficients in `nf`) on the algebraic number a (also in nf).

`GEN FpX_FpC_nfpoleval(GEN nf, GEN pol, GEN a, GEN p)` evaluate the `FpX pol` on the algebraic number a (also in nf).

The following three functions implement trivial functionality akin to Euclidean division for which we currently have no real use. Of course, even if the number field is actually Euclidean, these do not in general implement a true Euclidean division.

`GEN nfdiveuc(GEN nf, GEN a, GEN b)` returns the algebraic integer closest to x/y. Functionally identical to `ground( nfdiv(nf,x,y) )`.

`GEN nfdivrem(GEN nf, GEN a, GEN b)` returns the vector $[q, r]$, where

```
q = nfdiveuc(nf, a,b);
r = nfadd(nf, a,nfmul(nf,q,gneg(b)));    \\ or r = nfmod(nf,a,b);
```

`GEN nfmod(GEN nf, GEN a, GEN b)` returns r such that

```
q = nfdiveuc(nf, a,b);
r = nfadd(nf, a, nfmul(nf,q, gneg(b)));
```

`GEN nf_to_scalar_or_basis(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its basis representation (`nfalgtobasis`). Shallow function.

`GEN nf_to_scalar_or_alg(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its lifted `t_POLMOD` representation (lifted `nfbasistoalg`). Shallow function.

`GEN RgX_to_nfX(GEN nf, GEN x)` let x be a `t_POL` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new polynomial. Shallow function.

`GEN RgM_to_nfM(GEN nf, GEN x)` let x be a `t_MAT` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new matrix. Shallow function.

`GEN RgC_to_nfC(GEN nf, GEN x)` let x be a `t_COL` or `t_VEC` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new `t_COL`. Shallow function.

Unsafe routines. The following routines assume that their `nf` argument is a true *nf* (e.g. a *bnf* is not allowed) and their argument are restricted in various ways, see the precise description below.

`GEN nfinvmodideal(GEN nf, GEN x, GEN A)` given an algebraic integer x and a non-zero integral ideal A in HNF, returns a y such that $xy \equiv 1$ modulo A.

`GEN nfpowmodideal(GEN nf, GEN x, GEN n, GEN ideal)` given an algebraic integer x, an integer n, and a non-zero integral ideal A in HNF, returns an algebraic integer congruent to x^n modulo A.

`GEN nfmuli(GEN nf, GEN x, GEN y)` returns $x \times y$ assuming that both x and y are either `t_INT`s or `ZV`s of the correct dimension.

`GEN nfsqri(GEN nf, GEN x)` returns x^2 assuming that x is a `t_INT` or a `ZV` of the correct dimension.

`GEN nfC_nf_mul(GEN nf, GEN v, GEN x)` given a `t_VEC` or `t_COL` v of elements of K in `t_INT`, `t_FRAC` or `t_COL` form, multiply it by the element x (arbitrary form). This is faster than multiplying coordinatewise since pre-computations related to x (computing the multiplication table) are done only once. The components of the result are in most cases `t_COL`s but are allowed to be `t_INT`s or `t_FRAC`s. Shallow function.

`GEN zk_multable(GEN nf, GEN x)` given a `ZC` x (implicitly representing an algebraic integer), returns the `ZM` giving the multiplication table by x. Shallow function (the first column of the result points to the same data as x).

`GEN zk_scalar_or_multable(GEN nf, GEN x)` given a `t_INT` or `ZC` x, returns a `t_INT` equal to x if the latter is a scalar (`t_INT` or `ZV_isscalar`(x) is 1) and `zk_multable`(nf, x) otherwise. Shallow function.

The following routines implement multiplication in a commutative R-algebra, generated by $(e_1 = 1, \dots, e_n)$, and given by a multiplication table M: elements in the algebra are n-dimensional `t_COL`s, and the matrix M is such that for all $1 \le i, j \le n$, its column with index $(i-1)n + j$, say (c_k), gives $e_i \cdot e_j = \sum c_k e_k$. It is assumed that e_1 is the neutral element for the multiplication (a convenient optimization, true in practice for all multiplications we needed to implement). If x has any other type than `t_COL` where an algebra element is expected, it is understood as xe_1.

`GEN multable(GEN M, GEN x)` given a column vector x, representing the quantity $\sum_{i=1}^{N} x_i e_i$, returns the multiplication table by x. Shallow function.

`GEN ei_multable(GEN M, long i)` returns the multiplication table by the i-th basis element e_i. Shallow function.

`GEN tablemul(GEN M, GEN x, GEN y)` returns $x \cdot y$.

`GEN tablesqr(GEN M, GEN x)` returns x^2.

`GEN tablemul_ei(GEN M, GEN x, long i)` returns $x \cdot e_i$.

`GEN tablemul_ei_ej(GEN M, long i, long j)` returns $e_i \cdot e_j$.

`GEN tablemulvec(GEN M, GEN x, GEN v)` given a vector v of elements in the algebra, returns the $x \cdot v[i]$.

12.1.9 Elements in factored form.

Computational algebraic theory performs extensively linear algebra on $\mathbf{Z}$-modules with a natural multiplicative structure (K^*, fractional ideals in K, $\mathbf{Z}_K^*$, ideal class group), thereby raising elements to horrendously large powers. A seemingly innocuous elementary linear algebra operation like $C_i \leftarrow C_i - 10000C_1$ involves raising entries in C_1 to the 10000-th power. Understandably, it is often more efficient to keep elements in factored form rather than expand every such expression. A *factorization matrix* (or *famat*) is a two column matrix, the first column containing *elements* (arbitrary objects which may be repeated in the column), and the second one contains *exponents* (`t_INT`s, allowed to be 0). By abuse of notation, the empty matrix `cgetg(1, t_MAT)` is recognized as the trivial factorization (no element, no exponent).

Even though we think of a *famat* with columns g and e as one meaningful object when fully expanded as $\prod g[i]^{e[i]}$, *famat*s are basically about concatenating information to keep track of linear algebra: the objects stored in a *famat* need not be operation-compatible, they will not even be compared to each other (with one exception: `famat_reduce`). Multiplying two *famat*s just concatenates their elements and exponents columns. In a context where a *famat* is expected, an object x which is not of type `t_MAT` will be treated as the factorization x^1. The following functions all return *famat*s:

`GEN famat_mul(GEN f, GEN g)` f, g are *famat*, or objects whose type is *not* `t_MAT` (understood as f^1 or g^1). Returns fg. The empty factorization is the neutral element for *famat* multiplication.

`GEN famat_mul_shallow(GEN f, GEN g)` f, g are *famat*, returns fg. Shallow function.

`GEN famat_pow(GEN f, GEN n)` n is a `t_INT`. If f is a `t_MAT`, assume it is a *famat* and return f^n (multiplies the exponent column by n). Otherwise, understand it as an element and returns the 1-line *famat* f^n.

`GEN famat_sqr(GEN f)` returns f^2.

`GEN famat_inv(GEN f)` returns f^{-1}.

`GEN famat_inv_shallow(GEN f)` shallow version of `famat_inv`.

`GEN to_famat(GEN x, GEN k)` given an element x and an exponent k, returns the *famat* x^k.

`GEN to_famat_shallow(GEN x, GEN k)` same, as a shallow function.

Note that it is trivial to break up a *famat* into its two constituent columns: `gel(f,1)` and `gel(f,2)` are the elements and exponents respectively. Conversely, `mkmat2` builds a (shallow) *famat* from two `t_COL`s of the same length.

The last two functions makes an assumption about the elements: they must be regular algebraic numbers (not *famat*s) over a given number field:

`GEN famat_reduce(GEN f)` given a *famat* f, returns a *famat* g without repeated elements or 0 exponents, such that the expanded forms of f and g would be equal. Shallow function.

`GEN ZM_famat_limit(GEN f, GEN limit)` given a *famat* f with `t_INT` entries, returns a *famat* g with all factors larger than `limit` multiplied out as the last entry (with exponent 1).

`GEN famat_to_nf(GEN nf, GEN f)` You normally never want to do this! This is a simplified form of `nffactorback`, where we do not check the user input for consistency.

The description of `famat_to_nf` says that you do not want to use this function. Then how do we recover genuine number field elements? Well, in most cases, we do not need to: most of the functions useful in this context accept *famat*s as inputs, for instance `nfsign`, `nfsign_arch`, `ideallog` and `bnfisunit`. Otherwise, we can generally make good use of a quotient operation (modulo a fixed conductor, modulo ℓ-th powers); see the end of Section 12.1.19.

Caveat. Receiving a *famat* input, `bnfisunit` assumes that it is an algebraic integer, since this is expensive to check, and normally easy to ensure from the user's side; do not feed it ridiculous inputs.

`GEN famatsmall_reduce(GEN f)` as `famat_reduce`, but for exponents given by a `t_VECSMALL`.

12.1.10 Ideal arithmetic.

Conversion to HNF.

`GEN idealhnf(GEN nf, GEN x)` returns the HNF of the ideal defined by x: x may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is complicated, but useful to reduce general modules to the canonical form once in a while:

• if strictly less than $N = [K : Q]$ generators are given, x is the $\mathbf{Z}_K$-module they generate,

• if N or more are given, it is assumed that they form a $\mathbf{Z}$-basis (that the matrix has maximal rank N). This acts as `mathnf` since the $\mathbf{Z}_K$-module structure is (taken for granted hence) not taken into account in this case.

Extended ideals are also accepted, their principal part being discarded.

`GEN idealhnf0(GEN nf, GEN x, GEN y)` returns the HNF of the ideal generated by the two algebraic numbers x and y.

The following low-level functions underlie the above two: they all assume that `nf` is a true *nf* and perform no type checks:

`GEN idealhnf_principal(GEN nf, GEN x)` returns the ideal generated by the algebraic number x.

`GEN idealhnf_shallow(GEN nf, GEN x)` is `idealhnf` except that the result may not be suitable for `gerepile`: if x is already in HNF, we return x, not a copy!

`GEN idealhnf_two(GEN nf, GEN v)` assuming $a = v[1]$ is a non-zero `t_INT` and $b = v[2]$ is an algebraic integer, possibly given in regular representation by a `t_MAT` (the multiplication table by b, see `zk_multable`), returns the HNF of $a\mathbf{Z}_K + b\mathbf{Z}_K$.

Operations.

The basic ideal routines accept all `nf`s (*nf*, *bnf*, *bnr*) and ideals in any form, including extended ideals, and return ideals in HNF, or an extended ideal when that makes sense:

`GEN idealadd(GEN nf, GEN x, GEN y)` returns $x + y$.

`GEN idealdiv(GEN nf, GEN x, GEN y)` returns x/y. Returns an extended ideal if x or y is an extended ideal.

`GEN idealmul(GEN nf, GEN x, GEN y)` returns xy. Returns an extended ideal if x or y is an extended ideal.

`GEN idealsqr(GEN nf, GEN x)` returns x^2. Returns an extended ideal if x is an extended ideal.

`GEN idealinv(GEN nf, GEN x)` returns x^{-1}. Returns an extended ideal if x is an extended ideal.

`GEN idealpow(GEN nf, GEN x, GEN n)` returns x^n. Returns an extended ideal if x is an extended ideal.

`GEN idealpows(GEN nf, GEN ideal, long n)` returns x^n. Returns an extended ideal if x is an extended ideal.

`GEN idealmulred(GEN nf, GEN x, GEN y)` returns an extended ideal equal to xy.

`GEN idealpowred(GEN nf, GEN x, GEN n)` returns an extended ideal equal to x^n.

More specialized routines suffer from various restrictions:

`GEN idealdivexact(GEN nf, GEN x, GEN y)` returns x/y, assuming that the quotient is an integral ideal. Much faster than `idealdiv` when the norm of the quotient is small compared to Nx. Strips the principal parts if either x or y is an extended ideal.

`GEN idealdivpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns $x\mathfrak{p}^{-n}$, assuming x is an ideal in HNF, and `pr` a *prid* associated to $\mathfrak{p}$. Not suitable for `gerepileupto` since it returns x when $n = 0$.

`GEN idealmulpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns $x\mathfrak{p}^n$, assuming x is an ideal in HNF, and `pr` a *prid* associated to $\mathfrak{p}$. Not suitable for `gerepileupto` since it retunrs x when $n = 0$.

`GEN idealprodprime(GEN nf, GEN P)` given a list P of prime ideals in *prid* form, return their product.

`GEN idealmul_HNF(GEN nf, GEN x, GEN y)` returns xy, assuming than `nf` is a true *nf*, x is an integral ideal in HNF and y is an integral ideal in HNF or precompiled form (see below). For

maximal speed, the second ideal y may be given in precompiled form $y = [a, b]$, where a is a non-zero `t_INT` and b is an algebraic integer in regular representation (a `t_MAT` giving the multiplication table by the fixed element): very useful when many ideals x are going to be multiplied by the same ideal y. This essentially reduces each ideal multiplication to an $N \times N$ matrix multiplication followed by a $N \times 2N$ modular HNF reduction (modulo $xy \cap \mathbf{Z}$).

Approximation.

`GEN idealaddtoone(GEN nf, GEN A, GEN B)` given to coprime integer ideals A, B, returns $[a, b]$ with $a \in A$, $b \in B$, such that $a + b = 1$ The result is reduced mod AB, so a, b will be small.

`GEN idealaddtoone_i(GEN nf, GEN A, GEN B)` as `idealaddtoone` except that `nf` must be a true *nf*, and only a is returned.

`GEN hnfmerge_get_1(GEN A, GEN B)` given two square upper HNF integral matrices A, B of the same dimension $n > 0$, return a in the image of A such that $1 - a$ is in the image of B. (By abuse of notation we denote 1 the column vector $[1, 0, \ldots, 0]$.) If such an a does not exist, return `NULL`. This is the function underlying `idealaddtoone`.

`GEN idealaddmultoone(GEN nf, GEN v)` given a list of n (globally) coprime integer ideals ($v[i]$) returns an n-dimensional vector a such that $a[i] \in v[i]$ and $\sum a[i] = 1$. If $[K : \mathbf{Q}] = N$, this routine computes the HNF reduction (with $Gl_{nN}(\mathbf{Z})$ base change) of an $N \times nN$ matrix; so it is well worth pruning "useless" ideals from the list (as long as the ideals remain globally coprime).

`GEN idealappr(GEN nf, GEN x)` given a fractional ideal x, returns an algebraic number α such that $v(x) = v(\alpha)$ for all valuations such that $v(x) > 0$, and $v(\alpha) \geq 0$ at all others.

`GEN idealapprfact(GEN nf, GEN fx)` same as `idealappr`, x being given in factored form, as after `fx = idealfactor(nf,x)`, except that we allow 0 exponents in the factorization. Returns an algebraic number α such that $v(x) = v(\alpha)$ for all valuations associated to the prime ideal decomposition of x, and $v(\alpha) \geq 0$ at all others.

`GEN idealcoprime(GEN nf, GEN x, GEN y)`. Given 2 integral ideals x and y, returns an algebraic number α such that αx is an integral ideal coprime to y.

`GEN idealcoprimefact(GEN nf, GEN x, GEN fy)` same as `idealcoprime`, except that y is given in factored form, as from `idealfactor`.

`GEN idealchinese(GEN nf, GEN x, GEN y)` x being a prime ideal factorization (i.e. a 2 by 2 matrix whose first column contain prime ideals, and the second column integral exponents), y a vector of elements in *nf* indexed by the ideals in x, computes an element b such that $v_{\wp}(b - y_{\wp}) \geq v_{\wp}(x)$ for all prime ideals in x and $v_{\wp}(b) \geq 0$ for all other $\wp$.

12.1.11 Maximal ideals.

The PARI structure associated to maximal ideals is a *prid* (for *pr*ime *id*eal), usually produced by `idealprimedec` and `idealfactor`. In this section, we describe the format; other sections will deal with their daily use.

A *prid* associated to a maximal ideal `p` stores the following data: the underlying rational prime p, the ramification degree $e \geq 1$, the residue field degree $f \geq 1$, a p-uniformizer π with valuation 1 at `p` and valuation 0 at all other primes dividing p and a rescaled "anti-uniformizer" τ used to compute valuations. This τ is an algebraic integer such that τ/p has valuation -1 at `p` and valuation 0 at all other primes dividing p; in particular, the valuation of $x \in \mathbf{Z}_K$ is positive if and only if the algebraic integer $x\tau$ is divisible by p (easy to check for elements in `t_COL` form).

The following functions are shallow and return directly components of the *prid* `pr`:

`GEN pr_get_p(GEN pr)` returns p. Shallow function.

`GEN pr_get_gen(GEN pr)` returns π. Shallow function.

`long pr_get_e(GEN pr)` returns e.

`long pr_get_f(GEN pr)` returns f.

`GEN pr_get_tau(GEN pr)` returns `zk_scalar_or_multable`(nf, τ), which is the `t_INT` 1 iff p is inert, and a `ZM` otherwise. Shallow function.

`int pr_is_inert(GEN pr)` returns 1 if p is inert, 0 otherwise.

`GEN pr_norm(GEN pr)` returns the norm p^f of the maximal ideal.

12.1.12 Reducing modulo maximal ideals.

`GEN nfmodprinit(GEN nf, GEN pr)` returns an abstract `modpr` structure, associated to reduction modulo the maximal ideal `pr`, in `idealprimedec` format. From this data we can quickly project any `pr`-integral number field element to the residue field. This function is almost useless in library mode, we rather use:

`GEN nf_to_Fq_init(GEN nf, GEN *ppr, GEN *pT, GEN *pp)` concrete version of `nfmodprinit`: `nf` and `*ppr` are the inputs, the return value is a `modpr` and `*ppr`, `*pT` and `*pp` are set as side effects.

The input `*ppr` is either a maximal ideal or already a `modpr` (in which case it is replaced by the underlying maximal ideal). The residue field is realized as $\mathbf{F}_p[X]/(T)$ for some monic $T \in \mathbf{F}_p[X]$, and we set `*pT` to T and `*pp` to p. Set $T =$ `NULL` if the prime has degree 1 and the residue field is $\mathbf{F}_p$.

In short, this receives (or initializes) a `modpr` structure, and extracts from it T, p and `p`.

`GEN nf_to_Fq(GEN nf, GEN x, GEN modpr)` returns an `Fq` congruent to x modulo the maximal ideal associated to `modpr`. The output is canonical: all elements in a given residue class are represented by the same `Fq`.

`GEN Fq_to_nf(GEN x, GEN modpr)` returns an `nf` element lifting the residue field element x, either a `t_INT` or an algebraic integer in `algtobasis` format.

`GEN modpr_genFq(GEN modpr)` Returns an `nf` element whose image by `nf_to_Fq` is $X \pmod T$, if $\deg T > 1$, else 1.

`GEN zkmodprinit(GEN nf, GEN pr)` as `nfmodprinit`, but we assume we will only reduce algebraic integers, hence do not initialize data allowing to remove denominators. More precisely, we can in fact still handle an x whose rational denominator is not 0 in the residue field (i.e. if the valuation of x is non-negative at all primes dividing p).

`GEN zk_to_Fq_init(GEN nf, GEN *pr, GEN *T, GEN *p)` as `nf_to_Fq_init`, able to reduce only p-integral elements.

`GEN zk_to_Fq(GEN x, GEN modpr)` as `nf_to_Fq`, for a p-integral x.

`GEN nfM_to_FqM(GEN M, GEN nf,GEN modpr)` reduces a matrix of `nf` elements to the residue field; returns an `FqM`.

`GEN FqM_to_nfM(GEN M, GEN modpr)` lifts an `FqM` to a matrix of `nf` elements.

`GEN nfV_to_FqV(GEN A, GEN nf,GEN modpr)` reduces a vector of `nf` elements to the residue field; returns an `FqV` with the same type as `A` (`t_VEC` or `t_COL`).

`GEN FqV_to_nfV(GEN A, GEN modpr)` lifts an `FqV` to a vector of `nf` elements (same type as `A`).

`GEN nfX_to_FqX(GEN Q, GEN nf,GEN modpr)` reduces a polynomial with `nf` coefficients to the residue field; returns an `FqX`.

`GEN FqX_to_nfX(GEN Q, GEN modpr)` lifts an `FqX` to a polynomial with coefficients in `nf`.

12.1.13 Valuations.

`long nfval(GEN nf, GEN x, GEN P)` return $v_P(x)$

Unsafe functions. assume *nf* is a genuine `nf` structure, that P, Q are `prid`.

`long ZC_nfval(GEN nf, GEN x, GEN P)` returns $v_P(x)$, assuming x is a `ZC`, representing a non-zero algebraic integer.

`long ZC_nfvalrem(GEN nf, GEN x, GEN pr, GEN *newx)` returns $v = v_P(x)$, assuming x is a `ZC`, representing a non-zero algebraic integer, and sets `*newx` to $x\tau^v$ which is an algebraic integer coprime to p.

`int ZC_prdvd(GEN nf, GEN x, GEN P)` returns 1 is P divides x and 0 otherwise. Assumes that x is a `ZC`, representing an algebraic integer. Faster than computing $v_P(x)$.

`int pr_equal(GEN nf, GEN P, GEN Q)` returns 1 is P and Q represent the same maximal ideal: they must lie above the same p and share the same e, f invariants, but the p-uniformizer and τ element may differ. Returns 0 otherwise.

12.1.14 Signatures.

"Signs" of the real embeddings of number field element are represented in additive notation, using the standard identification $(\mathbf{Z}/2\mathbf{Z}, +) \to (\{-1, 1\}, \times)$, $s \mapsto (-1)^s$.

With respect to a fixed `nf` structure, a selection of real places (a divisor at infinity) is normally given as a `t_VECSMALL` of indices of the roots `nf.roots` of the defining polynomial for the number field. For compatibility reasons, in particular under GP, the (obsolete) `vec01` form is also accepted: a `t_VEC` with `gen_0` or `gen_1` entries.

The following internal functions go back and forth between the two representations for the Archimedean part of divisors (GP: 0/1 vectors, library: list of indices):

`GEN vec01_to_indices(GEN v)` given a `t_VEC` v with `t_INT` entries equal to 0 or 1, return as a `t_VECSMALL` the list of indices i such that $v[i] = 1$. If v is already a `t_VECSMALL`, return it (not suitable for `gerepile` in this case).

`GEN indices_to_vec01(GEN p, long n)` return the 0/1 vector of length n with ones exactly at the positions $p[1], p[2], \ldots$

`GEN nfsign(GEN nf,GEN x)` x being a number field element and `nf` any form of number field, return the $0-1$-vector giving the signs of the r_1 real embeddings of x, as a `t_VECSMALL`. Linear algebra functions like `Flv_add_inplace` then allow keeping track of signs in series of multiplications.

If x is a `t_VEC` of number field elements, return the matrix whose columns are the signs of the $x[i]$.

`GEN nfsign_arch(GEN nf,GEN x,GEN arch) arch` being a list of distinct real places, either in `vec01` (`t_VEC` with `gen_0` or `gen_1` entries) or `indices` (`t_VECSMALL`) form (see `vec01_to_indices`), returns the signs of x at the corresponding places. This is the low-level function underlying `nfsign`.

`GEN nfsign_units(GEN bnf, GEN archp, int add_tu) archp` being a divisor at infinity in `indices` form (or `NULL` for the divisor including all real places), return the signs at `archp` of a system of fundamental units for the field, in the same order as `bnf.tufu` if `add_tu` is set; and in the same order as `bnf.fu` otherwise.

`GEN nfsign_from_logarch(GEN L, GEN invpi, GEN archp)` given L the vector of the $\log \sigma(x)$, where σ runs through the (real or complex) embeddings of some number field, `invpi` being a floating point approximation to $1/\pi$, and `archp` being a divisor at infinity in `indices` form, return the signs of x at the corresponding places. This is the low-level function underlying `nfsign_units`; the latter is actually a trivial wrapper `bnf` structures include the $\log \sigma(x)$ for a system of fundamental units of the field.

`GEN set_sign_mod_divisor(GEN nf, GEN x, GEN y, GEN module, GEN sarch)` let $f = f_0 f_\infty$ be the divisor represented by `module`, x, y two number field elements. Returns yt with $t = 1 \bmod^* f$ such that x and ty have the same signs at f_∞; if $x =$ `NULL`, make ty totally positive at f_∞. `sarch` is the output of `nfarchstar(nf, f0, finf)`.

`GEN nfarchstar(GEN nf, GEN f0, GEN finf)` for a divisor $f = f_0 f_\infty$ represented by the integral ideal `f0` in HNF and the `finf` in `indices` form, returns $(\mathbf{Z}_K/f_\infty)^*$ in a form suitable for computations mod f. More precisely, returns $[c, g, M]$, where $c = [2, \dots, 2]$ gives the cyclic structure of that group ($\#f_\infty$ copies of $\mathbf{Z}/2\mathbf{Z}$), g a minimal system of independent generators, which are furthermore congruent to 1 mod f_0 (no condition if $f_0 = \mathbf{Z}_K$), and M is the matrix of signs of the $g[i]$ at f_∞, which is square and invertible over $\mathbf{F}_2$.

`GEN idealprincipalunits(GEN nf, GEN pr, long e)` returns the multiplicative group $(1 + pr)/(1 + pr^e)$ as an abelian group. Faster than `idealstar` when the norm of pr is large, since it avoids (useless) work in the multiplicative group of the residue field.

12.1.15 Maximal order and discriminant.

A number field $K = \mathbf{Q}[X]/(T)$ is defined by a monic $T \in \mathbf{Z}[X]$. The low-level function computing a maximal order is

`void nfmaxord(nfmaxord_t *S, GEN T0, long flag)`, where the polynomial T_0 is squarefree with integer coefficients. Let K be the étale algebra $\mathbf{Q}[X]/(T_0)$ and let $T =$ `ZX_Q_normalize`(T_0), i.e. $T = CT_0(X/L)$ is monic and integral for some $C, Q \in \mathbf{Q}$.

The structure `nfmaxord_t` is initialized by the call; it has the following fields:

```
GEN T0, T, dT, dK; /* T0, T, discriminants of T and K */
GEN unscale; /* the integer L */
GEN index; /* index of power basis in maximal order */
GEN dTP, dTE; /* factorization of |dT|, primes / exponents */
GEN dKP, dKE; /* factorization of |dK|, primes / exponents */
GEN basis; /* Z-basis for maximal order of Q[X]/(T) */
```

The exponent vectors are `t_VECSMALL`. The primes in `dTP` and `dKP` are pseudoprimes, not proven primes. We recommend restricting to $T = T_0$, i.e. either to pass the input polynomial through `ZX_Q_normalize` *before* the call, or to forget about T_0 and go on with the polynomial T; otherwise

`unscale` $\neq 1$, all data is expressed in terms of $T \neq T_0$, and needs to be converted to T_0. For instance to convert the basis to $\mathbf{Q}[X]/(T_0)$:

```
RgXV_unscale(S.basis, S.unscale)
```

Instead of passing T, one can use the format $[T, listP]$ as in `nfbasis` or `nfinit`, which computes an order which is maximal at a set of primes, but need not be the maximal order.

The `flag` is an or-ed combination of the binary flags:

`nf_PARTIALFACT`: do not try to fully factor `dT` and only look for primes less than `primelimit`. In that case, the elements in `dTP` and `dKP` need not all be primes. But the resulting `dK`, `index` and `basis` are correct provided there exists no prime $p >$ `primelimit` such that p^2 divides the field discriminant `dK`. This flag is *deprecated*: the $[T, listP]$ is safer and more flexible.

`nf_ROUND2`: use the ROUND2 algorithm instead of the default ROUND4. This flag is *deprecated*: this algorithm is consistently slower.

T is the input polynomial (monic `ZX`). The format $[T, \texttt{listP}]$ is also recognized, where `listP` is as in `nfbasis` and is used to compute a local integral basis with respect to a specific set of primes.

`GEN indexpartial(GEN T, GEN dT)` T a monic separable `ZX`, `dT` is either `NULL` (no information) or a multiple of the discriminant of T. Let $K = \mathbf{Q}[X]/(T)$ and $\mathbf{Z}_K$ its maximal order. Returns a multiple of the exponent of the quotient group $\mathbf{Z}_K/(\mathbf{Z}[X]/(T))$. In other word, a *denominator* d such that $dx \in \mathbf{Z}[X]/(T)$ for all $x \in \mathbf{Z}_K$.

12.1.16 Computing in the class group.

We compute with arbitrary ideal representatives (in any of the various formats seen above), and call

`GEN bnfisprincipal0(GEN bnf, GEN x, long flag)`. The `bnf` structure already contains information about the class group in the form $\oplus_{i=1}^{n}(\mathbf{Z}/d_i\mathbf{Z})g_i$ for canonical integers d_i (with $d_n \mid \ldots \mid d_1$ all > 1) and essentially random generators g_i, which are ideals in HNF. We normally do not need the value of the g_i, only that they are fixed once and for all and that any (non-zero) fractional ideal x can be expressed uniquely as $x = (t)\prod_{i=1}^{n} g_i^{e_i}$, where $0 \leq e_i < d_i$, and (t) is some principal ideal. Computing e is straightforward, but t may be very expensive to obtain explicitly. The routine returns (possibly partial) information about the pair $[e, t]$, depending on `flag`, which is an or-ed combination of the following symbolic flags:

• `nf_GEN` tries to compute t. Returns $[e, t]$, with t an empty vector if the computation failed. This flag is normally useless in non-trivial situations since the next two serve analogous purposes in more efficient ways.

• `nf_GENMAT` tries to compute t in factored form, which is much more efficient than `nf_GEN` if the class group is moderately large; imagine a small ideal $x = (t)g^{10000}$: the norm of t has 10000 as many digits as the norm of g; do we want to see it as a vector of huge meaningless integers? The idea is to compute e first, which is easy, then compute (t) as $x\prod g_i^{-e_i}$ using successive `idealmulred`, where the ideal reduction extracts small principal ideals along the way, eventually raised to large powers because of the binary exponentiation technique; the point is to keep this principal part in factored *unexpanded* form. Returns $[e, t]$, with t an empty vector if the computation failed; this should be exceedingly rare, unless the initial accuracy to which `bnf` was computed was ridiculously low (and then `bnfinit` should not have succeeded either). Setting/unsetting `nf_GEN` has no effect when this flag is set.

• nf_GEN_IF_PRINCIPAL tries to compute t *only* if the ideal is principal ($e = 0$). Returns gen_0 if the ideal is not principal. Setting/unsetting nf_GEN has no effect when this flag is set, but setting/unsetting nf_GENMAT is possible.

• nf_FORCE in the above, insist on computing t, even if it requires recomputing a bnf from scratch. This is a last resort, and normally the accuracy of a bnf can be increased without trouble, but it may be that some algebraic information simply cannot be recovered from what we have: see bnfnewprec. It should be very rare, though.

In simple cases where you do not care about t, you may use

GEN isprincipal(GEN bnf, GEN x), which is a shortcut for bnfisprincipal0(bnf, x, 0).

The following low-level functions are often more useful:

GEN isprincipalfact(GEN bnf, GEN C, GEN L, GEN f, long flag) is about the same as bnfisprincipal0 applied to $C\prod L[i]^{f[i]}$, where the $L[i]$ are ideals, the $f[i]$ integers and C is either an ideal or NULL (omitted). Make sure to include nf_GENMAT in flag!

GEN isprincipalfact_or_fail(GEN bnf, GEN C, GEN L, GEN f) is for delicate cases, where we must be more clever than nf_FORCE (it is used when trying to increase the accuracy of a *bnf*, for instance). If performs

```
isprincipalfact(bnf,C, L, f, nf_GENMAT);
```

but if it fails to compute t, it just returns a t_INT, which is the estimated precision (in words, as usual) that would have been sufficient to complete the computation. The point is that nf_FORCE does exactly this internally, but goes on increasing the accuracy of the bnf, then discarding it, which is a major inefficiency if you intend to compute lots of discrete logs and have selected a precision which is just too low. (It is sometimes not so bad since most of the really expensive data is cached in bnf anyway, if all goes well.) With this function, the *caller* may decide to increase the accuracy using bnfnewprec (and keep the resulting bnf!), or avoid the computation altogether. In any case the decision can be taken at the place where it is most likely to be correct.

12.1.17 Floating point embeddings, the T_2 quadratic form.

We assume the *nf* is a true nf structure, associated to a number field K of degree n and signature (r_1, r_2). We saw that

GEN nf_get_M(GEN nf) returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K, so that if v is an n-th dimensional t_COL representing the element $\sum_{i=1}^{n} v[i]w_i$ of K, then RgM_RgC_mul(M,v) represents the embeddings of v. Its first r_1 components are real numbers (t_INT, t_FRAC or t_REAL, usually the latter), and the last r_2 are complex numbers (usually of t_COMPLEX, but not necessarily for embeddings of rational numbers).

GEN embed_T2(GEN x, long r1) assuming x is the vector of floating point embeddings of some algebraic number v, i.e.

```
x = RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v));
```

returns $T_2(v)$. If the floating point embeddings themselves are not needed, but only the values of T_2, it is more efficient to restrict to real arithmetic and use

```
gnorml2( RgM_RgC_mul(nf_get_G(nf), algtobasis(nf,v)));
```

GEN embednorm_T2(GEN x, long r1) analogous to embed_T2, applied to the gnorm of the floating point embeddings. Assuming that

```
x = gnorm( RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v)) );
```

returns $T_2(v)$.

`GEN embed_roots(GEN z, long r1)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return the $r_1 + 2r_2$ roots of its characteristic polynomial. Shallow function.

`GEN embed_disc(GEN z, long r1, long prec)` given a vector z of $r_1 + r_2$ complex embeddings of the algebraic number v, return a floating point approximation of the discriminant of its characteristic polynomial as a `t_REAL` of precision `prec`.

`GEN embed_norm(GEN x, long r1)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return (a floating point approximation of) the norm of v.

12.1.18 Ideal reduction, low level.

In the following routines *nf* is a true `nf`, associated to a number field K of degree n:

`GEN nf_get_Gtwist(GEN nf, GEN v)` assuming v is a `t_VECSMALL` with $r_1 + r_2$ entries, let

$$||x||_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i}\varepsilon_i|\sigma_i(x)|^2,$$

where as usual the σ_i are the (real and) complex embeddings and $\varepsilon_i = 1$, resp. 2, for a real, resp. complex place. This is a twisted variant of the T_2 quadratic form, the standard Euclidean form on $K \otimes \mathbf{R}$. In applications, only the relative size of the v_i will matter.

Let $G_v \in M_n(\mathbf{R})$ be a square matrix such that if $x \in K$ is represented by the column vector X in terms of the fixed $\mathbf{Z}$-basis of $\mathbf{Z}_K$ in *nf*, then

$$||x||_v^2 = {}^t(G_vX) \cdot G_vX.$$

(This is a kind of Cholesky decomposition.) This function returns a rescaled copy of G_v, rounded to nearest integers, specifically `RM_round_maxrank`(G_v). Suitable for `gerepileupto`, but does not collect garbage.

`GEN nf_get_Gtwist1(GEN nf, long i)`. Simple special case. Returns the twisted G matrix associated to the vector v whose entries are all 0 except the i-th one, which is equal to 10.

`GEN idealpseudomin(GEN x, GEN G)`. Let x, G be two `ZM`s, such that the product Gx is well-defined. This returns a "small" integral linear combinations of the columns of x, given by the LLL-algorithm applied to the lattice Gx. Suitable for `gerepileupto`, but does not collect garbage.

In applications, x is an integral ideal, G approximates a Cholesky form for the T_2 quadratic form as returned by `nf_get_Gtwist`, and we return a small element a in the lattice (x, T_2). This is used to implement `idealred`.

`GEN idealpseudomin_nonscalar(GEN x, GEN G)`. As `idealpseudomin`, but we insist of returning a non-scalar a (`ZV_isscalar` is false), if the dimension of x is > 1.

In the interpretation where x defines an integral ideal on a fixed $\mathbf{Z}_K$ basis whose first element is 1, this means that a is not rational.

`GEN idealred_elt(GEN nf, GEN x)` shortcut for

```
idealpseudomin(x, nf_get_roundG(nf))
```

12.1.19 Ideal reduction, high level.

Given an ideal x this means finding a "simpler" ideal in the same ideal class. The public GP function is of course available

`GEN idealred0(GEN nf, GEN x, GEN v)` finds a small $a \in x$ and returns the primitive part of $x/(a)$, as an ideal in HNF. What "small" means depends on the parameter v, see the GP description. More precisely, a is returned by `idealpseudomin`(x, G), where G is `nf_get_Gtwist`(nf, v) for $v \neq$ `NULL` and `nf_get_roundG`(nf) otherwise.

Usually one sets $v =$ `NULL` to obtain an element of small T_2 norm in x:

`GEN idealred(GEN nf, GEN x)` is a shortcut for `idealred0(nf,x,NULL)`.

The function `idealred` remains complicated to use: in order not to lose information x must be an extended ideal, otherwise the value of a is lost. There is a subtlety here: the principal ideal (a) is easy to recover, but a itself is an instance of the principal ideal problem which is very difficult given only an *nf* (once a *bnf* structure is available, `bnfisprincipal0` will recover it). It is in general simpler to use directly `idealred_elt`.

`GEN idealmoddivisor(GEN bnr, GEN x)` A proof-of-concept implementation, useless in practice. If `bnr` is associated to some modulus f, returns a "small" ideal in the same class as x in the ray class group modulo f. The reason why this is useless is that using extended ideals with principal part in a computation, there is a simple way to reduce them: simply reduce the generator of the principal part in $(\mathbf{Z}_K/f)^*$.

`GEN famat_to_nf_moddivisor(GEN nf, GEN g, GEN e, GEN bid)` given a true *nf* associated to a number field K, a *bid* structure associated to a modulus f, and an algebraic number in factored form $\prod g[i]^{e[i]}$, such that $(g[i], f) = 1$ for all i, returns a small element in $\mathbf{Z}_K$ congruent to it mod f. Note that if f contains places at infinity, this includes sign conditions at the specified places.

A simpler case when the conductor has no place at infinity:

`GEN famat_to_nf_modideal_coprime(GEN nf, GEN g, GEN e, GEN f, GEN expo)` as above except that the ideal f is now integral in HNF (no need for a full *bid*), and we pass the exponent of the group $(\mathbf{Z}_K/f)^*$ as `expo`; any multiple will also do, at the expense of efficiency. Of course if a *bid* for f is available, if is easy to extract f and the exact value of `expo` from it (the latter is the first elementary divisor in the group structure). A useful trick: if you set `expo` to *any* positive integer, the result is correct up to `expo`-th powers, hence exact if `expo` is a multiple of the exponent; this is useful when trying to decide whether an element is a square in a residue field for instance! (take `expo`$= 2$).

What to do when the $g[i]$ are not coprime to f, but only $\prod g[i]^{e[i]}$ is? Then the situation is more complicated, and we advise to solve it one prime divisor of f at a time. Let v the valuation associated to a maximal ideal `pr` and assume $v(f) = k > 0$:

`GEN famat_makecoprime(GEN nf, GEN g, GEN e, GEN pr, GEN prk, GEN expo)` returns an element in $(\mathbf{Z}_K/\mathtt{pr}^k)^*$ congruent to the product $\prod g[i]^{e[i]}$, assumed to be globally coprime to f. As above, `expo` is any positive multiple of the exponent of $(\mathbf{Z}_K/\mathtt{pr}^k)^*$, for instance $(Nv-1)p^{k-1}$, if p is the underlying rational prime. You may use other values of `expo` (see the useful trick in `famat_to_nf_modideal_coprime`).

12.1.20 Class field theory.

Under GP, a class-field theoretic description of a number field is given by a triple A, B, C, where the defining set $[A, B, C]$ can have any of the following forms: $[bnr]$, $[bnr, subgroup]$, $[bnf, modulus]$, $[bnf, modulus, subgroup]$. You can still use directly all of (`libpari`'s routines implementing) GP's functions as described in Chapter 3, but they are often awkward in the context of `libpari` programming. In particular, it does not make much sense to always input a triple A, B, C because of the fringe $[bnf, modulus, subgroup]$. The first routine to call, is thus

`GEN Buchray(GEN bnf, GEN mod, long flag)` initializes a *bnr* structure from `bnf` and modulus `mod`. `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (if omitted, do not return a *bnr*, only the ray class group as an abelian group). In fact, a single value of `flag` actually makes sense: `nf_GEN | nf_INIT` to initialize a proper *bnr*: removing `nf_GEN` saves very little time, but the corresponding crippled *bnr* structure will raise errors in most class field theoretic functions. Possibly also 0 to quickly compute the ray class group structure; `bnrclassno` is faster if we only need the *order* of the ray class group.

Now we have a proper *bnr* encoding a `bnf` and a modulus, we no longer need the $[bnf, modulus]$ and $[bnf, modulus, subgroup]$ forms, which would internally call `Buchray` anyway. Recall that a subgroup H is given by a matrix in HNF, whose column express generators of H on the fixed generators of the ray class group that stored in our *bnr*. You may also code the trivial subgroup by `NULL`.

`GEN bnrconductor(GEN bnr, GEN H, long flag)` see the documentation of the GP function.

`long bnrisconductor(GEN bnr, GEN H)` returns 1 is the class field defined by the subgroup H (of the ray class group mod f coded in `bnr`) has conductor f. Returns 0 otherwise.

`GEN bnrdisc(GEN bnr, GEN H, long flag)` returns the discriminant and signature of the class field defined by `bnr` and H. See the description of the GP function for details. *flag* is an or-ed combination of the flags `rnf_REL` (output relative data) and `rnf_COND` (return 0 unless the modulus is the conductor).

`GEN bnrsurjection(GEN BNR, GEN bnr)` `BNR` and `bnr` defined over the same field K, for moduli F and f with $F \mid f$, returns the matrix of the canonical surjection $\mathrm{Cl}_K(F) \to \mathrm{Cl}_K(f)$ (giving the image of the fixed ray class group generators of `BNR` in terms of the ones in `bnr`). `BNR` must include the ray class group generators.

`GEN ABC_to_bnr(GEN A, GEN B, GEN C, GEN *H, int addgen)` This is a quick conversion function designed to go from the too general (inefficient) A, B, C form to the preferred *bnr*, H form for class fields. Given A, B, C as explained above (omitted entries coded by `NULL`), return the associated *bnr*, and set H to the associated subgroup. If `addgen` is 1, make sure that if the *bnr* needed to be computed, then it contains generators.

12.1.21 Relative equations, Galois conjugates.

`GEN rnfequationall(GEN A, GEN B, long *pk, GEN *pLPRS)` A is either an *nf* type (corresponding to a number field K) or an irreducible `ZX` defining a number field K. B is an irreducible polynomial in $K[X]$. Returns an absolute equation C (over $\mathbf{Q}$) for the number field $K[X]/(B)$. C is the characteristic polynomial of $b + ka$ for some roots a of A and b of B, and k is a small rational integer. Set `*pk` to k.

If `pLPRS` is not `NULL` set it to $[h_0, h_1]$, $h_i \in \mathbf{Q}[X]$, where $h_0 + h_1 Y$ is the last non-constant polynomial in the pseudo-Euclidean remainder sequence associated to $A(Y)$ and $B(X - kY)$, leading

to $C = \mathrm{Res}_Y(A(Y), B(Y - kX))$. In particular $a := -h_0/h_1$ is a root of A in $\mathbf{Q}[X]/(C)$, and $X - ka$ is a root of B.

`GEN nf_rnfeq(GEN A, GEN B)` wrapper around `rnfequationall` to allow mapping $K \to L$ (`eltup`) and converting elements of L between absolute and relative form (`reltoabs`, `abstorel`), *without* computing a full *rnf* structure, which is useful if the relative integral basis is not required. In fact, since A may be a `t_POL` or an *nf*, the integral basis of the base field is not needed either. The return value is the same as `rnf_get_map`. Shallow function.

`GEN nf_rnfeqsimple(GEN nf, GEN relpol)` as `nf_rnfeq` except some fields are omitted, so that only the `abstorel` operation is supported. Shallow function.

`GEN eltabstorel(GEN rnfeq, GEN x)` `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltabstorel` is more robust), `nf_rnfeq` or `nf_rnfeqsimple`, return x as an element of L/K, i.e. as a `t_POLMOD` with `t_POLMOD` coefficients. Shallow function.

`GEN eltabstorel_lift(GEN rnfeq, GEN x)` same as `eltabstorel`, except that x is returned in partially lifted form, i.e. as a `t_POL` with `t_POLMOD` coefficients.

`GEN eltreltoabs(GEN rnfeq, GEN x)` `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltreltoabs` is more robust) or `nf_rnfeq`, return x in absolute form.

`void nf_nfzk(GEN nf, GEN rnfeq, GEN *zknf, GEN *czknf)` `rnfeq` as given by `nf_rnfeq`, `nf` a true *nf* structure, set `*zknf` and `*czknf` to a suitable representation of `nf.zk` allowing quick computation of the map $K \to L$ by the function `nfeltup`, *without* computing a full *rnf* structure, which is useful if the relative integral basis is not required. The computed values are the same as in `rnf_get_nfzk`. Shallow function.

`GEN nfeltup(GEN nf, GEN x, GEN zknf, GEN czknf)` `zknf` and `czknf` are initialized by `nf_nfzk` or `rnf_get_nfzk` (but in this case `rnfeltup` is more robust); `nf` is a true *nf* structure for K, returns $x \in K$ as a (lifted) element of L, in absolute form.

`GEN Rg_nffix(const char *f, GEN T, GEN c, int lift)` given a `ZX` T and a "coefficient" c supposedly belonging to $\mathbf{Q}[y]/(T)$, check whether this is a the case and return a cleaned up version of c. The string f is the calling function name, used to report errors.

This means that c must be one of `t_INT`, `t_FRAC`, `t_POL` in the variable y with rational coefficients, or `t_POLMOD` modulo T which lift to a rational `t_POL` as above. The cleanup consists in the following improvements:

- `t_POL` coefficients are reduced modulo T.
- `t_POL` and `t_POLMOD` belonging to $\mathbf{Q}$ are converted to rationals, `t_INT` or `t_FRAC`.
- if `lift` is non-zero, convert `t_POLMOD` to `t_POL`, and otherwise convert `t_POL` to `t_POLMOD`s modulo T.

`GEN RgX_nffix(const char *f, GEN T, GEN P, int lift)` check whether P is a polynomials with coefficients in the number field defined by the absolute equation $T(y) = 0$, where T is a `ZX` and returns a cleaned up version of P. This checks whether P is indeed a `t_POL` with variable compatible with coefficients in $\mathbf{Q}[y]/(T)$, i.e.

```
varncmp(varn(P), varn(T)) < 0
```

and applies `Rg_nffix` to each coefficient.

`GEN RgV_nffix(const char *f, GEN T, GEN P, int lift)` as `RgX_nffix` for a vector of coefficients.

`GEN polmod_nffix(const char *f, GEN rnf, GEN x, int lift)` given a `t_POLMOD` x supposedly defining an element of *rnf*, check this and perform `Rg_nffix` cleanups.

`GEN polmod_nffix2(const char *f, GEN T, GEN P, GEN x, int lift)` as `polmod_nffix`, where the relative extension is explicitly defined as $L = (\mathbf{Q}[y]/(T))[x]/(P)$, instead of by an `rnf` structure.

`long numberofconjugates(GEN T, long pinit)` returns a quick multiple for the number of $\mathbf{Q}$-automorphism of the (integral, monic) `t_POL` T, from modular factorizations, starting from prime `pinit` (you can set it to 2). This upper bounds often coincides with the actual number of conjugates. Of course, you should use `nfgaloisconj` to be sure.

12.1.22 Obsolete routines.

Still provided for backward compatibility, but should not be used in new programs. They will eventually disappear.

`GEN zidealstar(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN)`

`GEN zidealstarinit(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_INIT)`

`GEN zidealstarinitgen(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN|nf_INIT)`

`GEN buchimag(GEN D, GEN c1, GEN c2, GEN gCO)` short for

```
Buchquad(D,gtodouble(c1),gtodouble(c2), /*ignored*/0)
```

`GEN buchreal(GEN D, GEN gsens, GEN c1, GEN c2, GEN RELSUP, long prec)` short for

```
Buchquad(D,gtodouble(c1),gtodouble(c2), prec)
```

The following use a naming scheme which is error-prone and not easily extensible; besides, they compute generators as per `nf_GEN` and not `nf_GENMAT`. Don't use them:

`GEN isprincipalforce(GEN bnf,GEN x)`

`GEN isprincipalgen(GEN bnf, GEN x)`

`GEN isprincipalgenforce(GEN bnf, GEN x)`

`GEN isprincipalraygen(GEN bnr, GEN x)`, use `bnrisprincipal`.

Variants on `polred`: use `polredbest`.

`GEN factoredpolred(GEN x, GEN fa)`

`GEN factoredpolred2(GEN x, GEN fa)`

`GEN smallpolred(GEN x)`

`GEN smallpolred2(GEN x)`, use `Polred`.

`GEN polredabs(GEN x)`

`GEN polredabs2(GEN x)`

`GEN polredabsall(GEN x, long flun)`

`nfmaxord` wrappers implementing the old `nfbasis` interface: use the $[T, listP$ format.

`GEN polred0(GEN x, long flag, GEN p)`

`GEN nfbasis0(GEN x,long flag,GEN p)`

`GEN nfdisc0(GEN x,long flag, GEN p)`

`GEN factorpadic0(GEN f,GEN p,long r,long flag)`

Superseded by `bnrdisc`:

`GEN discrayabs(GEN bnr,GEN subgroup)`

`GEN discrayabscond(GEN bnr,GEN subgroup)`

`GEN discrayrel(GEN bnr,GEN subgroup)`

`GEN discrayrelcond(GEN bnr,GEN subgroup)`

Superseded by `bnrdisclist0`:

`GEN discrayabslist(GEN bnf,GEN listes)`

`GEN discrayabslistarch(GEN bnf, GEN arch, long bound)`

`GEN discrayabslistlong(GEN bnf, long bound)`

12.2 Galois extensions of Q.

This section describes the data structure output by the function `galoisinit`. This will be called a `gal` structure in the following.

12.2.1 Extracting info from a `gal` structure.

The functions below expect a `gal` structure and are shallow. See the documentation of `galoisinit` for the meaning of the member functions.

`GEN gal_get_pol(GEN gal)` returns `gal.pol`

`GEN gal_get_p(GEN gal)` returns `gal.p`

`GEN gal_get_e(GEN gal)` returns the integer e such that `gal.mod==gal.p^e`.

`GEN gal_get_mod(GEN gal)` returns `gal.mod`.

`GEN gal_get_roots(GEN gal)` returns `gal.roots`.

`GEN gal_get_invvdm(GEN gal)` `gal[4]`.

`GEN gal_get_den(GEN gal)` return `gal[5]`.

`GEN gal_get_group(GEN gal)` returns `gal.group`.

`GEN gal_get_gen(GEN gal)` returns `gal.gen`.

`GEN gal_get_orders(GEN gal)` returns `gal.orders`.

12.2.2 Miscellaneous functions.

`GEN nfgaloismatrix(GEN nf, GEN s)` returns the `ZM` associated to the automorphism s, seen as a linear operator expressend on the number field integer basis. This allows to use

```
M = nfgaloismatrix(nf, s);
sx = ZM_ZC_mul(M, x);   /* or RgM_RgC_mul(M, x) if x is not integral */
```

instead of

```
sx = nfgaloisapply(nf, s, x);
```

for an algebraic integer x.

12.3 Quadratic number fields and quadratic forms.

12.3.1 Checks.

`void check_quaddisc(GEN x, long *s, long *mod4, const char *f)` checks whether the `GEN` x is a quadratic discriminant (`t_INT`, not a square, congruent to $0, 1$ modulo 4), and raise an exception otherwise. Set `*s` to the sign of x and `*mod4` to x modulo 4 (0 or 1).

`void check_quaddisc_real(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is positive.

`void check_quaddisc_imag(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is negative.

12.3.2 `t_QFI`, `t_QFR`.

`GEN qfi(GEN x, GEN y, GEN z)` creates the `t_QFI` (x, y, z).

`GEN qfr(GEN x, GEN y, GEN z, GEN d)` creates the `t_QFR` (x, y, z) with distance component d.

`GEN qfr_1(GEN q)` given a `t_QFR` q, return the unit form q^0.

`GEN qfi_1(GEN q)` given a `t_QFI` q, return the unit form q^0.

12.3.2.1 Composition.

`GEN qficomp(GEN x, GEN y)` compose the two `t_QFI` x and y, then reduce the result. This is the same as `gmul(x,y)`.

`GEN qfrcomp(GEN x, GEN y)` compose the two `t_QFR` x and y, then reduce the result. This is the same as `gmul(x,y)`.

`GEN qfisqr(GEN x)` as `qficomp(x,y)`.

`GEN qfrsqr(GEN x)` as `qfrcomp(x,y)`.

Same as above, *without* reducing the result:

`GEN qficompraw(GEN x, GEN y)`

`GEN qfrcompraw(GEN x, GEN y)`

`GEN qfisqrraw(GEN x)`

`GEN qfrsqrraw(GEN x)`

`GEN qfbcompraw(GEN x, GEN y)` compose two `t_QFI`s or two `t_QFR`s, without reduce the result.

12.3.2.2 Powering.

`GEN powgi(GEN x, GEN n)` computes x^n (will work for many more types than `t_QFI` and `t_QFR`, of course). Reduce the result.

`GEN qfrpow(GEN x, GEN n)` computes x^n for a `t_QFR` x, reducing along the way. If the distance component is initially 0, leave it alone; otherwise update it.

`GEN qfbpowraw(GEN x, long n)` compute x^n (pure composition, no reduction), for a `t_QFI` or `t_QFR` x.

`GEN qfipowraw(GEN x, long n)` as `qfbpowraw`, for a `t_QFI` x.

`GEN qfrpowraw(GEN x, long n)` as `qfbpowraw`, for a `t_QFR` x.

12.3.2.3 Solve, Cornacchia.

The following functions underly `qfbsolve`; p denotes a prime number.

`GEN qfisolvep(GEN Q, GEN p)` solves $Q(x, y) = p$ over the integers, for a `t_QFI` Q. Return `gen_0` if there are no solutions.

`GEN qfrsolvep(GEN Q, GEN p)` solves $Q(x, y) = p$ over the integers, for a `t_QFR` Q. Return `gen_0` if there are no solutions.

`long cornacchia(GEN d, GEN p, GEN *px, GEN *py)` solves $x^2 + dy^2 = p$ over the integers, where $d > 0$. Return 1 if there is a solution (and store it in `*x` and `*y`), 0 otherwise.

`long cornacchia2(GEN d, GEN p, GEN *px, GEN *py)` as `cornacchia`, for the equation $x^2 + dy^2 = 4p$.

12.3.2.4 Prime forms.

`GEN primeform_u(GEN x, ulong p)` `t_QFI` whose first coefficient is the prime p.

`GEN primeform(GEN x, GEN p, long prec)`

12.3.3 Efficient real quadratic forms. Unfortunately, `t_QFR`s are very inefficient, and are only provided for backward compatibility.

• they do not contain needed quantities, which are thus constantly recomputed (the discriminant D, $\sqrt{D}$ and its integer part),

• the distance component is stored in logarithmic form, which involves computing one extra logarithm per operation. It is much more efficient to store its exponential, computed from ordinary multiplications and divisions (taking exponent overflow into account), and compute its logarithm at the very end.

Internally, we have two representations for real quadratic forms:

• `qfr3`, a container $[a, b, c]$ with at least 3 entries: the three coefficients; the idea is to ignore the distance component.

• `qfr5`, a container with at least 5 entries $[a, b, c, e, d]$: the three coefficients a `t_REAL` d and a `t_INT` e coding the distance component $2^{Ne}d$, in exponential form, for some large fixed N.

It is a feature that `qfr3` and `qfr5` have no specified length or type. It implies that a `qfr5` or `t_QFR` will do whenever a `qfr3` is expected. Routines using these objects all require a global context, provided by a `struct qfr_data *`:

```
struct qfr_data {
  GEN D;        /* discriminant, t_INT   */
  GEN sqrtD;    /* sqrt(D), t_REAL       */
  GEN isqrtD;   /* floor(sqrt(D)), t_INT */
};
```

`void qfr_data_init(GEN D, long prec, struct qfr_data *S)` given a discriminant $D > 0$, initialize S for computations at precision `prec` ($\sqrt{D}$ is computed to that initial accuracy).

All functions below are shallow, and not stack clean.

`GEN qfr3_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr3`, reducing the result.

`GEN qfr3_pow(GEN x, GEN n, struct qfr_data *S)` compute x^n, reducing along the way.

`GEN qfr3_red(GEN x, struct qfr_data *S)` reduce x.

`GEN qfr3_rho(GEN x, struct qfr_data *S)` perform one reduction step; `qfr3_red` just performs reduction steps until we hit a reduced form.

`GEN qfr3_to_qfr(GEN x, GEN d)` recover an ordinary `t_QFR` from the `qfr3` x, adding distance component d.

Before we explain `qfr5`, recall that it corresponds to an ideal, that reduction corresponds to multiplying by a principal ideal, and that the distance component is a clever way to keep track of these principal ideals. More precisely, reduction consists in a number of reduction steps, going from the form (a, b, c) to $\rho(a, b, c) = (c, -b \bmod 2c, *)$; the distance component is multiplied by (a floating point approximation to) $(b + \sqrt{D})/(b - \sqrt{D})$.

`GEN qfr5_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr5`, reducing the result, and updating the distance component.

`GEN qfr5_pow(GEN x, GEN n, struct qfr_data *S)` compute x^n, reducing along the way.

`GEN qfr5_red(GEN x, struct qfr_data *S)` reduce x.

`GEN qfr5_rho(GEN x, struct qfr_data *S)` perform one reduction step.

`GEN qfr5_dist(GEN e, GEN d, long prec)` decode the distance component from exponential (`qfr5`-specific) to logarithmic form (as in a `t_QFR`).

`GEN qfr_to_qfr5(GEN x, long prec)` convert a `t_QFR` to a `qfr5` with initial trivial distance component ($= 1$).

`GEN qfr5_to_qfr(GEN x, GEN d)`, assume x is a `qfr5` and d was the original distance component of some `t_QFR` that we converted using `qfr_to_qfr5` to perform efficiently a number of operations. Convert x to a `t_QFR` with the correct (logarithmic) distance component.

12.4 Linear algebra over Z.

12.4.1 Hermite and Smith Normal Forms.

`GEN ZM_hnf(GEN x)` returns the upper triangular Hermite Normal Form of the `ZM` x (removing 0 columns), using the `ZM_hnfall` algorithm. If you want the true HNF, use `ZM_hnfall(x, NULL, 0)`.

`GEN ZM_hnfmod(GEN x, GEN d)` returns the HNF of the `ZM` x (removing 0 columns), assuming the `t_INT` d is a multiple of the determinant of x. This is usually faster than `ZM_hnf` (and uses less memory) if the dimension is large, > 50 say.

`GEN ZM_hnfmodid(GEN x, GEN d)` returns the HNF of the matrix $(x \mid d\mathrm{Id})$ (removing 0 columns), for a `ZM` x and a `t_INT` d.

`GEN ZM_hnfmodall(GEN x, GEN d, long flag)` low-level function underlying the `ZM_hnfmod` variants. If `flag` is 0, calls `ZM_hnfmod(x,d)`; `flag` is an or-ed combination of:

• `hnf_MODID` call `ZM_hnfmodid` instead of `ZM_hnfmod`,

• `hnf_PART` return as soon as we obtain an upper triangular matrix, saving time. The pivots are non-negative and give the diagonal of the true HNF, but the entries to the right of the pivots need not be reduced, i.e. they may be large or negative.

• `hnf_CENTER` returns the centered HNF, where the entries to the right of a pivot p are centered residues in $[-p/2, p/2[$, hence smallest possible in absolute value, but possibly negative.

`GEN ZM_hnfall(GEN x, GEN *U, long remove)` returns the upper triangular HNF H of the `ZM` x; if U is not `NULL`, set if to the matrix U such that $xU = H$. If `remove` $= 0$, H is the true HNF, including 0 columns; if `remove` $= 1$, delete the 0 columns from H but do not update U accordingly (so that the integer kernel may still be recovered): we no longer have $xU = H$; if `remove` $= 2$, remove 0 columns from H and update U so that $xU = H$. The matrix U is square and invertible unless `remove` $= 2$.

This routine uses a naive algorithm which is potentially exponential in the dimension (due to coefficient explosion) but is fast in practice, although it may require lots of memory. The base change matrix U may be very large, when the kernel is large.

`GEN ZM_hnfperm(GEN A, GEN *ptU, GEN *ptperm)` returns the hnf $H = PAU$ of the matrix PA, where P is a suitable permutation matrix, and $U \in \mathrm{Gl}_n(\mathbf{Z})$. P is chosen so as to (heuristically) minimize the size of U; in this respect it is less efficient than `ZM_hnflll` but usually faster. Set `*ptU` to U and `*pterm` to a `t_VECSMALL` representing the row permutation associated to $P = (\delta_{i,\mathtt{perm}[i]}$. If `ptU` is set to `NULL`, U is not computed, saving some time; although useless, setting `ptperm` to `NULL` is also allowed.

`GEN ZM_hnflll(GEN x, GEN *U, int remove)` returns the HNF H of the `ZM` x; if U is not `NULL`, set if to the matrix U such that $xU = H$. The meaning of `remove` is the same as in `ZM_hnfall`.

This routine uses the LLL variant of Havas, Majewski and Mathews, which is polynomial time, but rather slow in practice because it uses an exact LLL over the integers instead of a floating point variant; it uses polynomial space but lots of memory is needed for large dimensions, say larger than 300. On the other hand, the base change matrix U is essentially optimally small with respect to the L_2 norm.

`GEN ZM_hnfcenter(GEN M)`. Given a `ZM` in HNF M, update it in place so that non-diagonal entries belong to a system of *centered* residues. Not suitable for gerepile.

Some direct applications: the following routines apply to upper triangular integral matrices; in practice, these come from HNF algorithms.

`GEN hnf_divscale(GEN A, GEN B,GEN t)` A an upper triangular `ZM`, B a `ZM`, t an integer, such that $C := tA^{-1}B$ is integral. Return C.

`GEN hnf_solve(GEN A, GEN B)` A a `ZM` in upper HNF (not necessarily square), B a `ZM` or `ZC`. Return $A^{-1}B$ if it is integral, and `NULL` if it is not.

`GEN hnf_invimage(GEN A, GEN b)` A a `ZM` in upper HNF (not necessarily square), b a `ZC`. Return $A^{-1}B$ if it is integral, and `NULL` if it is not.

`int hnfdivide(GEN A, GEN B)` A and B are two upper triangular `ZM`. Return 1 if $A^{-1}B$ is integral, and 0 otherwise.

Smith Normal Form.

`GEN ZM_snf(GEN x)` returns the Smith Normal Form (vector of elementary divisors) of the `ZM` x.

`GEN ZM_snfall(GEN x, GEN *U, GEN *V)` returns `ZM_smith(x)` and sets U and V to unimodular matrices such that $U\,x\,V = D$ (diagonal matrix of elementary divisors). Either (or both) U or V may be `NULL` in which case the corresponding matrix is not computed.

`GEN ZM_snfall_i(GEN x, GEN *U, GEN *V, int returnvec)` same as `ZM_snfall`, except that, depending on the value of `returnvec`, we either return a diagonal matrix (as in `ZM_snfall`, `returnvec` is 0) or a vector of elementary divisors (as in `ZM_snf`, `returnvec` is 1) .

`void ZM_snfclean(GEN d, GEN U, GEN V)` assuming d, U, V come from `d = ZM_snfall(x, &U, &V)`, where U or V may be `NULL`, cleans up the output in place. This means that elementary divisors equal to 1 are deleted and U, V are updated. The output is not suitable for `gerepileupto`.

`GEN ZM_snf_group(GEN H, GEN *U, GEN *Uinv)` this function computes data to go back and forth between an abelian group (of finite type) given by generators and relations, and its canonical SNF form. Given an abstract abelian group with generators $g = (g_1, \dots, g_n)$ and a vector $X = (x_i) \in \mathbf{Z}^n$, we write gX for the group element $\sum_i x_i g_i$; analogously if M is an $n \times r$ integer matrix gM is a vector containing r group elements. The group neutral element is 0; by abuse of notation, we still write 0 for a vector of group elements all equal to the neutral element. The input is a full relation matrix H among the generators, i.e. a `ZM` (not necessarily square) such that $gX = 0$ for some $X \in \mathbf{Z}^n$ if and only if X is in the integer image of H, so that the abelian group is isomorphic to $\mathbf{Z}^n/\mathrm{Im}H$. *The routine assumes that H is in HNF;* replace it by its HNF if it is not the case. (Of course this defines the same group.)

Let G a minimal system of generators in SNF for our abstract group: if the d_i are the elementary divisors $(\dots \mid d_2 \mid d_1)$, each G_i has either infinite order ($d_i = 0$) or order $d_i > 1$. Let D the matrix with diagonal (d_i), then

$$GD = 0, \quad G = gU_{\mathrm{inv}}, \quad g = GU,$$

for some integer matrices U and U_{inv}. Note that these are not even square in general; even if square, there is no guarantee that these are unimodular: they are chosen to have minimal entries given the known relations in the group and only satisfy $D \mid (UU_{\mathrm{inv}} - \mathrm{Id})$ and $H \mid (U_{\mathrm{inv}}U - \mathrm{Id})$.

The function returns the vector of elementary divisors (d_i); if `U` is not `NULL`, it is set to U; if `Uinv` is not `NULL` it is set to U_{inv}. The function is not memory clean.

The following 3 routines underly the various `matrixqz` variants. In all case the $m \times n$ `t_MAT` x is assumed to have rational (`t_INT` and `t_FRAC`) coefficients

`GEN QM_ImQ_hnf(GEN x)` returns an HNF basis for $\mathrm{Im}_{\mathbf{Q}} x \cap \mathbf{Z}^n$.

`GEN QM_ImZ_hnf(GEN x)` returns an HNF basis for $\mathrm{Im}_{\mathbf{Z}} x \cap \mathbf{Z}^n$.

`GEN QM_minors_coprime(GEN x, GEN D)`, assumes $m \geq n$, and returns a matrix in $M_{m,n}(\mathbf{Z})$ with the same $\mathbf{Q}$-image as x, such that the GCD of all $n \times n$ minors is coprime to D; if D is `NULL`, we want the GCD to be 1.

The following routines are simple wrappers around the above ones and are normally useless in library mode:

`GEN hnf(GEN x)` checks whether x is a `ZM`, then calls `ZM_hnf`. Normally useless in library mode.

`GEN hnfmod(GEN x, GEN d)` checks whether x is a `ZM`, then calls `ZM_hnfmod`. Normally useless in library mode.

`GEN hnfmodid(GEN x,GEN d)` checks whether x is a `ZM`, then calls `ZM_hnfmodid`. Normally useless in library mode.

`GEN hnfall(GEN x)` calls `ZM_hnfall(x, &U, 1)` and returns $[H, U]$. Normally useless in library mode.

`GEN hnflll(GEN x)` calls `ZM_hnflll(x, &U, 1)` and returns $[H, U]$. Normally useless in library mode.

`GEN hnfperm(GEN x)` calls `ZM_hnfperm(x, &U, &P)` and returns $[H, U, P]$. Normally useless in library mode.

`GEN smith(GEN x)` checks whether x is a `ZM`, then calls `ZM_smith`. Normally useless in library mode.

`GEN smithall(GEN x)` checks whether x is a `ZM`, then calls `ZM_smithall(x, &U, &V)` and returns $[U, V, D]$. Normally useless in library mode.

Some related functions over $K[X]$, K a field:

`GEN gsmith(GEN A)` the input matrix must be square, returns the elementary divisors.

`GEN gsmithall(GEN A)` the input matrix must be square, returns the $[U, V, D]$, D diagonal, such that $UAV = D$.

`GEN RgM_hnfall(GEN A, GEN *pB, long remove)` analogous to `ZM_hnfall`.

`GEN smithclean(GEN z)` cleanup the output of `smithall` or `gsmithall` (delete elementary divisors equal to 1, updating base change matrices).

12.4.2 The LLL algorithm.

The basic GP functions and their immediate variants are normally not very useful in library mode. We briefly list them here for completeness, see the documentation of `qflll` and `qflllgram` for details:

- `GEN qflll0(GEN x, long flag)`

`GEN lll(GEN x)` *flag*$= 0$

`GEN lllint(GEN x)` *flag*$= 1$

`GEN lllkerim(GEN x)` *flag*$= 4$

`GEN lllkerimgen(GEN x)` *flag*= 5

`GEN lllgen(GEN x)` *flag*= 8

- `GEN qflllgram0(GEN x, long flag)`

`GEN lllgram(GEN x)` *flag*= 0

`GEN lllgramint(GEN x)` *flag*= 1

`GEN lllgramkerim(GEN x)` *flag*= 4

`GEN lllgramkerimgen(GEN x)` *flag*= 5

`GEN lllgramgen(GEN x)` *flag*= 8

The basic workhorse underlying all integral and floating point LLLs is

`GEN ZM_lll(GEN x, double D, long flag)`, where x is a `ZM`; $D \in]1/4, 1[$ is the Lovász constant determining the frequency of swaps during the algorithm: a larger values means better guarantees for the basis (in principle smaller basis vectors) but longer running times (suggested value: $D = 0.99$).

Important. This function does not collect garbage and its output is not suitable for either `gerepile` or `gerepileupto`. We expect the caller to do something simple with the output (e.g. matrix multiplication), then collect garbage immediately.

`flag` is an or-ed combination of the following flags:

- `LLL_GRAM`. If set, the input matrix x is the Gram matrix ${}^t vv$ of some lattice vectors v.

- `LLL_INPLACE`. If unset, we return the base change matrix U, otherwise the transformed matrix xU or ${}^t UxU$ (`LLL_GRAM`). Implies `LLL_IM` (see below).

- `LLL_KEEP_FIRST`. The first vector in the output basis is the same one as was originally input. Provided this is a shortest non-zero vector of the lattice, the output basis is still LLL-reduced. This is used to reduce maximal orders of number fields with respect to the T_2 quadratic form, to ensure that the first vector in the output basis corresponds to 1 (which is a shortest vector).

The last three flags are mutually exclusive, either 0 or a single one must be set:

- `LLL_KER` If set, only return a kernel basis K (not LLL-reduced).

- `LLL_IM` If set, only return an LLL-reduced lattice basis T. (This is implied by `LLL_INPLACE`).

- `LLL_ALL` If set, returns a 2-component vector $[K, T]$ corresponding to both kernel and image.

`GEN lllfp(GEN x, double D, long flag)` is a variant for matrices with inexact entries: x is a matrix with real coefficients (types `t_INT`, `t_FRAC` and `t_REAL`), D and *flag* are as in `ZM_lll`. The matrix is rescaled, rounded to nearest integers, then fed to `ZM_lll`. The flag `LLL_INPLACE` is still accepted but less useful (it returns an LLL-reduced basis associated to rounded input, instead of an exact base change matrix).

`GEN ZM_lll_norms(GEN x, double D, long flag, GEN *ptB)` slightly more general version of `ZM_lll`, setting `*ptB` to a vector containing the squared norms of the Gram-Schmidt vectors (b_i^*) associated to the output basis (b_i), $b_i^* = b_i + \sum_{j<i} \mu_{i,j} b_j^*$.

`GEN lllintpartial_inplace(GEN x)` given a `ZM` x of maximal rank, returns a partially reduced basis (b_i) for the space spanned by the columns of x: $|b_i \pm b_j| \geq |b_i|$ for any two distinct basis vectors b_i, b_j. This is faster than the LLL algorithm, but produces much larger bases.

`GEN lllintpartial(GEN x)` as `lllintpartial_inplace`, but returns the base change matrix U from the canonical basis to the b_i, i.e. xU is the output of `lllintpartial_inplace`.

12.4.3 Reduction modulo matrices.

`GEN ZC_hnfremdiv(GEN x, GEN y, GEN *Q)` assuming y is an invertible `ZM` in HNF and x is a `ZC`, returns the `ZC` R equal to x mod y (whose i-th entry belongs to $[-y_{i,i}/2, y_{i,i}/2[$). Stack clean *unless* x is already reduced (in which case, returns x itself, not a copy). If Q is not `NULL`, set it to the `ZC` such that $x = yQ + R$.

`GEN ZM_hnfdivrem(GEN x, GEN y, GEN *Q)` reduce each column of the `ZM` x using `ZC_hnfremdiv`. If Q is not `NULL`, set it to the `ZM` such that $x = yQ + R$.

`GEN ZC_hnfrem(GEN x, GEN y)` alias for `ZC_hnfremdiv(x,y,NULL)`.

`GEN ZM_hnfrem(GEN x, GEN y)` alias for `ZM_hnfremdiv(x,y,NULL)`.

`GEN ZC_reducemodmatrix(GEN v, GEN y)` Let y be a ZM, not necessarily square, which is assumed to be LLL-reduced (otherwise, very poor reduction is expected). Size-reduces the ZC v modulo the $\mathbf{Z}$-module Y spanned by y : if the columns of y are denoted by $(y_1, \dots, y_{n-1})$, we return $y_n \equiv v$ modulo Y, such that the Gram-Schmidt coefficients $\mu_{n,j}$ are less than $1/2$ in absolute value for all $j < n$. In short, y_n is almost orthogonal to Y.

`GEN ZM_reducemodmatrix(GEN v, GEN y)` Let y be as in `ZC_reducemodmatrix`, and v be a ZM. This returns a matrix v which is congruent to v modulo the $\mathbf{Z}$-module spanned by y, whose columns are size-reduced. This is faster than repeatedly calling `ZC_reducemodmatrix` on the columns since most of the Gram-Schmidt coefficients can be reused.

`GEN ZC_reducemodlll(GEN v, GEN y)` Let y be an arbitrary ZM, LLL-reduce it then call `ZC_reducemodmatrix`.

`GEN ZM_reducemodlll(GEN v, GEN y)` Let y be an arbitrary ZM, LLL-reduce it then call `ZM_reducemodmatrix`.

Besides the above functions, which were specific to integral input, we also have:

`GEN reducemodinvertible(GEN x, GEN y)` y is an invertible matrix and x a `t_COL` or `t_MAT` of compatible dimension. Returns $x - y\lfloor y^{-1}x \rceil$, which has small entries and differs from x by an integral linear combination of the columns of y. Suitable for `gerepileupto`, but does not collect garbage.

`GEN closemodinvertible(GEN x, GEN y)` returns $x -$ `reducemodinvertible`(x, y), i.e. an integral linear combination of the columns of y, which is close to x.

`GEN reducemodlll(GEN x,GEN y)` LLL-reduce the non-singular `ZM` y and call `reducemodinvertible` to find a small representative of x mod $y\mathbf{Z}^n$. Suitable for `gerepileupto`, but does not collect garbage.

12.4.4 Miscellaneous.

`GEN RM_round_maxrank(GEN G)` given a matrix G with real floating point entries and independent columns, let G_e be the rescaled matrix 2^eG rounded to nearest integers, for $e \geq 0$. Finds a small e such that the rank of G_e is equal to the rank of G (its number of columns) and return G_e. This is useful as a preconditioning step to speed up LLL reductions, see `nf_get_Gtwist`. Suitable for `gerepileupto`, but does not collect garbage.

Chapter 13:
Technical Reference Guide for Elliptic curves and arithmetic geometry

This chapter is quite short, but is added as a placeholder, since we expect the library to expand in that direction.

13.1 Elliptic curves.

Elliptic curves are represented in the Weierstrass model

$$(E) : y^2z + a_1xyz + a_3yz = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

by the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$. Points in the projective plane are represented as follows: the point at infinity $(0 : 1 : 0)$ is coded as `[0]`, a finite point $(x : y : 1)$ outside the projective line at infinity $z = 0$ is coded as $[x, y]$. Note that other points at infinity than $(0 : 1 : 0)$ cannot be represented; this is harmless, since they do not belong to any of the elliptic curves E above.

Points on the curve are just projective points as described above, they are not tied to a curve in any way: the same point may be used in conjunction with different curves, provided it satisfies their equations (if it does not, the result is usually undefined). In particular, the point at infinity belongs to all elliptic curves.

As with `factor` for polynomial factorization, the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$ implicitly defines a base ring over which the curve is defined. Point coordinates must be operation-compatible with this base ring (`gadd`, `gmul`, `gdiv` involving them should not give errors).

13.1.1 Types of elliptic curves.

We call a 5-tuble as above an `ell5`; most functions require an `ell` structure, as returned by `ellinit`, which contains additional data (usually dynamically computed as needed), depending on the base field.

`GEN ellinit(GEN E, GEN D, long prec)`, returns an `ell` structure, associated to the elliptic curve E : either an `ell5`, a pair $[a_4, a_6]$ or a `t_STR` in Cremona's notation, e.g. `"11a1"`. The optional D (`NULL` to omit) describes the domain over which the curve is defined.

13.1.2 Type checking.

`void checkell(GEN e)` raise an error unless e is a *ell*.

`void checkell5(GEN e)` raise an error unless e is an *ell* or an *ell5*.

`void checkellpt(GEN z)` raise an error unless z is a point (either finite or at infinity).

`long ell_get_type(GEN e)` returns the domain type over which the curve is defined, one of

`t_ELL_Q` the field of rational numbers;

`t_ELL_Qp` the field of p-adic numbers, for some prime p;

`t_ELL_Fp` a prime finite field, base field elements are represented as `Fp` (`t_INT` reduced modulo p);

`t_ELL_Fq` a non-prime finite field (a prime finite field can also be represented by this subtype, but this is inefficient), base field elements are represented as `t_FFELT`;

`t_ELL_Rg` none of the above.

`void checkell_Fq(GEN e)` checks whether e is an `ell`, defined over a finite field (either prime or non-prime), raises `pari_err_TYPE` otherwise.

`void checkell_Q(GEN e)` checks whether e is an `ell`, defined over $\mathbf{Q}$, raises `pari_err_TYPE` otherwise.

`void checkell_Qp(GEN e)` checks whether e is an `ell`, defined over some $\mathbf{Q}_p$, raises `pari_err_TYPE` otherwise.

13.1.3 Extracting info from an `ell` structure.

These functions expect an `ell` argument. If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

13.1.3.1 All domains.

`GEN ell_get_a1(GEN e)`

`GEN ell_get_a2(GEN e)`

`GEN ell_get_a3(GEN e)`

`GEN ell_get_a4(GEN e)`

`GEN ell_get_a6(GEN e)`

`GEN ell_get_b2(GEN e)`

`GEN ell_get_b4(GEN e)`

`GEN ell_get_b6(GEN e)`

`GEN ell_get_b8(GEN e)`

`GEN ell_get_c4(GEN e)`

`GEN ell_get_c6(GEN e)`

`GEN ell_get_disc(GEN e)`

`GEN ell_get_j(GEN e)`

13.1.3.2 Curves over Q.

`GEN ellQ_get_N(GEN e)` returns the curve conductor

`void ellQ_get_Nfa(GEN e, GEN *N, GEN *faN)` sets N to the conductor and `faN` to its factorization

`long ellrootno_global(GEN e)` returns $[c, [c_{p_1}, \dots, c_{p_k}]]$, where the `t_INT` $c \in \{-1, 1\}$ is the global root number, and the c_{p_i} are the local root numbers at all prime divisors of the conductor, ordered as in `faN` above.

`GEN elldatagenerators(GEN E)` returns generators for $E(\mathbf{Q})$ extracted from Cremona's table.

`GEN ellanal_globalred(GEN e, GEN *v)` takes an *ell* over $\mathbf{Q}$ and returns a global minimal model E (in `ellinit` form, over $\mathbf{Q}$) for e suitable for analytic computations related to the curve L series:

it contains `ellglobalred` data, as well as global and local root numbers. If `v` is not `NULL`, set `*v` to the needed change of variable: `NULL` if e was already the standard minimal model, such that $E =$ `ellchangecurve(e,v)` otherwise. Compared to the direct use of `ellchangecurve` followed by `ellrootno`, this function avoids converting unneeded dynamic data and avoids potential memory leaks (the changed curve would have had to be deleted using `obj_free`). The original curve e is updated as well with the same information.

13.1.3.3 Curves over $\mathbf{Q}_p$.

`GEN ellQp_get_p(GEN E)` returns p

`long ellQp_get_prec(GEN E)` returns the default p-adic accuracy to which we must compute approximate results associated to E.

`GEN ellQp_get_zero(GEN x)` returns $O(p^n)$, where n is the default p-adic accuracy as above.

The following functions are only defined when E has multiplicative reduction (Tate curves):

`GEN ellQp_Tate_uniformization(GEN E, long prec)` returns a `t_VEC` containing $u^2, u, q, [a, b]$, at p-adic precision `prec`.

`GEN ellQp_u(GEN E, long prec)` returns u.

`GEN ellQp_u2(GEN E, long prec)` returns u^2.

`GEN ellQp_q(GEN E, long prec)` returns the Tate period q.

`GEN ellQp_ab(GEN E, long prec)` returns $[a, b]$.

`GEN ellQp_root(GEN E, long prec)` returns e_1.

13.1.3.4 Curves over a finite field $\mathbf{F}_q$.

`GEN ellff_get_p(GEN E)` returns the characteristic

`GEN ellff_get_field(GEN E)` returns p if $\mathbf{F}_q$ is a prime field, and a `t_FFELT` belonging to $\mathbf{F}_q$ otherwise.

`GEN ellff_get_card(GEN E)` returns $\#E(\mathbf{F}_q)$

`GEN ellff_get_gens(GEN E)` returns a minimal set of generators for $E(\mathbf{F}_q)$.

`GEN ellff_get_group(GEN E)` returns `ellgroup`(E).

`GEN ellff_get_o(GEN E)` returns $[d,$`factord`$]$, where d is the exponent of $E(\mathbf{F}_q)$.

`GEN ellff_get_a4a6(GEN E)` returns a canonical "short model" for E, and the corresponding change of variable $[u, r, s, t]$. For $p \neq 2, 3$, this is $[A_4, A_6, [u, r, s, t]]$, corresponding to $y^2 = x^3 + A_4x + A_6$, where $A_4 = -27c_4$, $A_6 = -54c_6$, $[u, r, s, t] = [6, 3b_2, 3a_1, 108a_3]$.

• If $p = 3$ and the curve is ordinary ($b_2 \neq 0$), this is $[[b_2], A_6, [1, v, -a_1, -a_3]]$, corresponding to

$$y^2 = x^3 + b_2x + A_6,$$

where $v = b_4/b_2$, $A_6 = b_6 - v(b_4 + v^2)$.

• If $p = 3$ and the curve is supersingular ($b_2 = 0$), this is $[-b_4, b_6, [1, 0, -a_1, -a_3]]$, corresponding to

$$y^2 = x^3 + 2b_4x + b_6.$$

• If $p = 2$ and the curve is ordinary ($a_1 \neq 0$), return $[A_2, A_6, [a_1^{-1}, da_1^{-2}, 0, (a_4 + d^2)a_1^{-1}]]$, corresponding to

$$y^2 + xy = x^3 + A_2x^2 + A_6,$$

where $d = a_3/a_1$, $a_1^2A_2 = (a_2 + d)$ and

$$a_1^6A_6 = d^3 + a_2d^2 + a_4d + a_6 + (a_4^2 + d^4)a_1^{-2}.$$

• If $p = 2$ and the curve is supersingular ($a_1 = 0$, $a_3 \neq 0$), return $[[a_3, A_4, 1/a_3], A_6, [1, a_2, 0, 0]]$, corresponding to

$$y^2 + a_3y = x^3 + A_4x + A_6,$$

where $A_4 = a_2^2 + a_4$, $A_6 = a_2a_4 + a_6$. The value $1/a_3$ is included in the vector since it is frequently needed in computations.

13.1.3.5 Curves over C. (This includes curves over **Q**!)

`long ellR_get_prec(GEN E)` returns the default accuracy to which we must compute approximate results associated to E.

`GEN ellR_ab(GEN E, long prec)` returns $[a, b]$

`GEN ellR_omega(GEN x, long prec)` returns periods $[\omega_1, \omega_2]$.

`GEN ellR_eta(GEN E, long prec)` returns quasi-periods $[\eta_1, \eta_2]$.

`GEN ellR_roots(GEN E, long prec)` returns $[e_1, e_2, e_3]$. If E is defined over **R**, then e_1 is real. If furthermore disc$E > 0$, then $e_1 > e_2 > e_3$.

`long ellR_get_sign(GEN E)` if E is defined over **R** returns the signe of its discriminant, otherwise return 0.

13.1.4 Points.

`int ell_is_inf(GEN z)` tests whether the point z is the point at infinity.

`GEN ellinf()` returns the point at infinity `[0]`.

13.1.5 Change of variables. `GEN ellchangeinvert(GEN w)` given a change of variables $w = [u, r, s, t]$, returns the inverse change of variables w', such that if $E' =$ `ellchangecurve(E, w)`, then $E =$ `ellchangecurve`(E, w').

13.1.6 Functions to handle elliptic curves over finite fields.

13.1.6.1 Tolerant routines.

`GEN ellap(GEN E, GEN p)` given a prime number p and an elliptic curve defined over **Q** or $\mathbf{Q}_p$ (assumed integral and minimal at p), computes the trace of Frobenius $a_p = p + 1 - \#E(\mathbf{F}_p)$. If E is defined over a non-prime finite field $\mathbf{F}_q$, ignore p and return $q + 1 - \#E(\mathbf{F}_q)$. When p is implied (E defined over $\mathbf{Q}_p$ or a finite field), p can be omitted (set to `NULL`).

`GEN ellsea(GEN E, GEN p, long s)` available if the `seadata` package is installed. This function returns $\#E(\mathbf{F}_p)$, using the Schoof-Elkies-Atkin algorithm; it is called by `ellap`: same conditions as above for E, except that `t_ELL_Fq` are not allowed. The extra flag `s`, if set to a non-zero value, causes the computation to return `gen_0` (an impossible cardinality) if one of the small primes $\ell > s$ divides the curve order. For cryptographic applications, where one is usually interested in curves of prime order, setting $s = 1$ efficiently weeds out most uninteresting curves; if curves of order a power of 2 times a prime are acceptable, set $s = 2$. There is no guarantee that the resulting cardinality is prime, only that it has no small prime divisor larger than s.

13.1.6.2 Curves defined a non-prime finite field. In this subsection, we assume that `ell_get_type`(E) is `t_ELL_Fq`. (As noted above, a curve defined over $\mathbf{Z}/p\mathbf{Z}$ can be represented as a `t_ELL_Fq`.)

`GEN FF_ellmul(GEN E, GEN P, GEN n)` returns $[n]P$ where n is an integer and P is a point on the curve E.

`GEN FF_ellrandom(GEN E)` returns a random point in $E(\mathbf{F}_q)$. This function never returns the point at infinity, unless this is the only point on the curve.

`GEN FF_ellorder(GEN E, GEN P, GEN o)` returns the order of the point P, where o is a multiple of the order of P, or its factorization.

`GEN FF_ellcard(GEN E)` returns $\#E(\mathbf{F}_q)$.

`GEN FF_ellgens(GEN E)` returns the generators of the group $E(\mathbf{F}_q)$.

`GEN FF_elllog(GEN E, GEN P, GEN G, GEN o)` Let `G` be a point of order `o`, return e such that $[e]P = G$. If e does not exists, the result is undefined.

`GEN FF_ellgroup(GEN E)` returns the Abelian group $E(\mathbf{F}_q)$ in the form $[h, \texttt{cyc}, \texttt{gen}]$.

`GEN FF_ellweilpairing(GEN E, GEN P, GEN Q, GEN m)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN FF_elltatepairing(GEN E, GEN P, GEN Q, GEN m)` returns the Tate pairing of P and Q, where $[m]P = 0$.

13.2 Arithmetic on elliptic curve over a finite field in simple form.

The functions in this section no longer operate on elliptic curve structures, as seen up to now. They are used to implement those higher-level functions without using cached information and thus require suitable explicitly enumerated data.

13.2.1 Helper functions.

`GEN elltrace_extension(GEN t, long n, GEN q)` Let E some elliptic curve over $\mathbf{F}_q$ such that the trace of the Frobenius is t, returns the trace of the Frobenius over $\mathbf{F}_q^n$.

13.2.2 Elliptic curves over $\mathbf{F}_p$, $p > 3$.

Let p a prime number and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p$. A `FpE` is a point of $E(\mathbf{F}_p)$. Since an affine point and a_4 determine an unique $a6$, most functions do not take a_6 as an argument. A `FpE` is either the point at infinity (`ellinf()`) or a FpV whith two components. The parameters a_4 and a_6 are given as `t_INT`s when required.

`GEN Fp_ellj(GEN a4, GEN a6, GEN p)` returns the j-invariant of the curve E.

`GEN Fp_ellcard(GEN a4, GEN a6, GEN p)` returns the cardinal of the group $E(\mathbf{F}_p)$.

`GEN Fp_ellcard_SEA(GEN a4, GEN a6, GEN p, long s)` same as `ellsea` when only $[a_4, a_6]$ are given.

`GEN Fq_ellcard_SEA(GEN a4, GEN a6, GEN q, GEN T, GEN p, long s)` same as `ellsea` when only $[a_4, a_6]$ are given, over $\mathbf{F}_p[t]/(T)$. Assume $p \neq 2, 3$.

`GEN Fp_ffellcard(GEN a4, GEN a6, GEN q, long n, GEN p)` returns the cardinal of the group $E(\mathbf{F}_q)$ where $q = p^n$.

`GEN Fp_ellgroup(GEN a4, GEN a6, GEN N, GEN p, GEN *pt_m)` returns the group structure D of the group $E(\mathbf{F}_p)$, which is assumed to be of order N and set $*pt_m = m$.

`GEN Fp_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN p)` returns generators of the group $E(\mathbf{F}_p)$ with the base change `ch` (see `FpE_changepoint`), where D and m are as returned by `Fp_ellgroup`.

`GEN Fp_elldivpol(GEN a4, GEN a6, long n, GEN p)` returns the n-division polynomial of the elliptic curve E.

13.2.3 FpE.

`GEN FpE_add(GEN P, GEN Q, GEN a4, GEN p)` returns the sum $P+Q$ in the group $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$, for any value of a_6 compatible with the points given.

`GEN FpE_sub(GEN P, GEN Q, GEN a4, GEN p)` returns $P - Q$.

`GEN FpE_dbl(GEN P, GEN a4, GEN p)` returns $2.P$.

`GEN FpE_neg(GEN P, GEN p)` returns $-P$.

`GEN FpE_mul(GEN P, GEN n, GEN a4, GEN p)` return $n.P$.

`GEN FpE_changepoint(GEN P, GEN m, GEN a4, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FpV`).

`GEN FpE_changepointinv(GEN P, GEN m, GEN a4, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FpV`).

`GEN random_FpE(GEN a4, GEN a6, GEN p)` returns a random point on $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$.

`GEN FpE_order(GEN P, GEN o, GEN a4, GEN p)` returns the order of P in the group $E(\mathbf{F}_p)$, where o is a multiple of the order of P, or its factorization.

`GEN FpE_log(GEN P, GEN G, GEN o, GEN a4, GEN p)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN FpE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN p)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN FpE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN p)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN FpE_to_mod(GEN P, GEN p)` returns P as a vector of `t_INTMOD`s.

`GEN RgE_to_FpE(GEN P, GEN p)` returns the `FpE` obtained by applying `Rg_to_Fp` coefficientwise.

13.2.4 `Fle`. Let p be a prime `ulong`, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are `ulong`. A `Fle` is either the point at infinity (`ellinf()`), or a `Flv` with two components.

`long Fl_elltrace(ulong a4, ulong a6, ulong p)` returns the trace t of the Frobenius of $E(\mathbf{F}_p)$. The cardinal of $E(\mathbf{F}_p)$ is thus $p + 1 - t$, which might not fit in a `ulong`.

`GEN Fle_add(GEN P, GEN Q, ulong a4, ulong p)`

`GEN Fle_dbl(GEN P, ulong a4, ulong p)`

`GEN Fle_sub(GEN P, GEN Q, ulong a4, ulong p)`

`GEN Fle_mul(GEN P, GEN n, ulong a4, ulong p)`

`GEN Fle_mulu(GEN P, ulong n, ulong a4, ulong p)`

`GEN Fle_order(GEN P, GEN o, ulong a4, ulong p)`

`GEN random_Fle(ulong a4, ulong a6, ulong p)`

13.2.5 Elliptic curves over $\mathbf{F}_{2^n}$. Let T be an irreducible `F2x` and E the elliptic curve given by either the equation $E : y^2 + x * y = x^3 + a_2x^2 + a_6$, where a_2, a_6 are `F2x` in $\mathbf{F}_2[X]/(T)$ (ordinary case) or $E : y^2 + a_3 * y = x^3 + a_4x + a_6$, where a_3, a_4, a_6 are `F2x` in $\mathbf{F}_2[X]/(T)$ (supersingular case).

A `F2xqE` is a point of $E(\mathbf{F}_2[X]/(T))$. In the supersingular case, the parameter `a2` is actually the `t_VEC` $[a_3, a_4, a_3^{-1}]$.

`GEN F2xq_ellcard(GEN a2, GEN a6, GEN T)` Return the order of the group $E(\mathbf{F}_2[X]/(T))$.

`GEN F2xq_ellgroup(GEN a2, GEN a6, GEN N, GEN T, GEN *pt_m)` Return the group structure D of the group $E(\mathbf{F}_2[X]/(T))$, which is assumed to be of order N and set $*pt_m = m$.

`GEN F2xq_ellgens(GEN a2, GEN a6, GEN ch, GEN D, GEN m, GEN T)` Returns generators of the group $E(\mathbf{F}_2[X]/(T))$ with the base change `ch` (see `F2xqE_changepoint`), where D and m are as returned by `F2xq_ellgroup`.

13.2.6 `F2xqE`.

`GEN F2xqE_changepoint(GEN P, GEN m, GEN a2, GEN T)` returns the image Q of the point P on the curve $E : y^2 + x * y = x^3 + a_2x^2 + a_6$ by the coordinate change m (which is a `F2xqV`).

`GEN F2xqE_changepointinv(GEN P, GEN m, GEN a2, GEN T)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `F2xqV`).

`GEN F2xqE_add(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_sub(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_dbl(GEN P, GEN a2, GEN T)`

`GEN F2xqE_neg(GEN P, GEN a2, GEN T)`

`GEN F2xqE_mul(GEN P, GEN n, GEN a2, GEN T)`

`GEN random_F2xqE(GEN a2, GEN a6, GEN T)`

`GEN F2xqE_order(GEN P, GEN o, GEN a2, GEN T)` returns the order of P in the group $E(\mathbf{F}_2[X]/(T))$, where o is a multiple of the order of P, or its factorization.

`GEN F2xqE_log(GEN P, GEN G, GEN o, GEN a2, GEN T)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN F2xqE_tatepairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN F2xqE_weilpairing(GEN Q, GEN Q, GEN m, GEN a2, GEN T)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN RgE_to_F2xqE(GEN P, GEN T)` returns the `F2xqE` obtained by applying `Rg_to_F2xq` coefficientwise.

13.2.7 Elliptic curves over $\mathbf{F}_q$, small characteristic $p > 2$. Let p be a prime `ulong`, T an irreducible `Flx` mod p, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are `Flx` in $\mathbf{F}_p[X]/(T)$. A `FlxqE` is a point of $E(\mathbf{F}_p[X]/(T))$.

In the special case $p = 3$, ordinary elliptic curves ($j(E) \neq 0$) cannot be represented as above, but admit a model $E : y^2 = x^3 + a_2x^2 + a_6$ with a_2 and a_6 being `Flx` in $\mathbf{F}_3[X]/(T)$. In that case, the parameter `a2` is actually stored as a `t_VEC`, $[a_2]$, to avoid ambiguities.

`GEN Flxq_ellj(GEN a4, GEN a6, GEN T, ulong p)` returns the j-invariant of the curve E.

`GEN Flxq_ellcard(GEN a4, GEN a6, GEN T, ulong p)` returns the order of $E(\mathbf{F}_p[X]/(T))$.

`GEN Flxq_ellgroup(GEN a4, GEN a6, GEN N, GEN T, ulong p, GEN *pt_m)` returns the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and set $*pt_m = m$.

`GEN Flxq_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, ulong p)` returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FlxqE_changepoint`), where D and m are as returned by `Flxq_ellgroup`.

13.2.8 `FlxqE`.

`GEN FlxqE_changepoint(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_changepointinv(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_add(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_sub(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_dbl(GEN P, GEN a4, GEN T, ulong p)`

`GEN FlxqE_neg(GEN P, GEN T, ulong p)`

`GEN FlxqE_mul(GEN P, GEN n, GEN a4, GEN T, ulong p)`

`GEN random_FlxqE(GEN a4, GEN a6, GEN T, ulong p)`

`GEN FlxqE_order(GEN P, GEN o, GEN a4, GEN T, ulong p)` returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P, or its factorization.

`GEN FlxqE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, ulong p)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN FlxqE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN FlxqE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN RgE_to_FlxqE(GEN P, GEN T, ulong p)` returns the `FlxqE` obtained by applying `Rg_to_Flxq` coefficientwise.

13.2.9 Elliptic curves over $\mathbf{F}_q$, large characteristic .

Let p be a prime number, T an irreducible polynomial mod p, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$ with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$. A `FpXQE` is a point of $E(\mathbf{F}_p[X]/(T))$.

`GEN FpXQ_ellj(GEN a4, GEN a6, GEN T, GEN p)` returns the j-invariant of the curve E.

`GEN FpXQ_ellcard(GEN a4, GEN a6, GEN T, GEN p)` returns the order of $E(\mathbf{F}_p[X]/(T))$.

`GEN FpXQ_ellgroup(GEN a4, GEN a6, GEN N, GEN T, GEN p, GEN *pt_m)` Return the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and set $*pt_m = m$.

`GEN FpXQ_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, GEN p)` Returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FpXQE_changepoint`), where D and m are as returned by `FpXQ_ellgroup`.

`GEN FpXQ_elldivpol(GEN a4, GEN a6, long n, GEN T, GEN p)` returns the n-division polynomial of the elliptic curve E.

`GEN Fq_elldivpolmod(GEN a4,GEN a6, long n, GEN h, GEN T, GEN p)` returns the n-division polynomial of the elliptic curve E modulo the polynomial h.

13.2.10 `FpXQE`.

`GEN FpXQE_changepoint(GEN P, GEN m, GEN a4, GEN T, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FpXQV`).

`GEN FpXQE_changepointinv(GEN P, GEN m, GEN a4, GEN T, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FpXQV`).

`GEN FpXQE_add(GEN P, GEN Q, GEN a4, GEN T, GEN p)`

`GEN FpXQE_sub(GEN P, GEN Q, GEN a4, GEN T, GEN p)`

`GEN FpXQE_dbl(GEN P, GEN a4, GEN T, GEN p)`

`GEN FpXQE_neg(GEN P, GEN T, GEN p)`

`GEN FpXQE_mul(GEN P, GEN n, GEN a4, GEN T, GEN p)`

`GEN random_FpXQE(GEN a4, GEN a6, GEN T, GEN p)`

`GEN FpXQE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, GEN p)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN FpXQE_order(GEN P, GEN o, GEN a4, GEN T, GEN p)` returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P, or its factorization.

`GEN FpXQE_tatepairing(GEN P,GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN FpXQE_weilpairing(GEN P,GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN RgE_to_FpXQE(GEN P, GEN T, GEN p)` returns the `FpXQE` obtained by applying `Rg_to_FpXQ` coefficientwise.

13.3 Other curves.

The following functions deal with hyperelliptic curves in weighted projective space $\mathbf{P}_{(1,d,1)}$, with coordinates (x, y, z) and a model of the form $y^2 = T(x, z)$, where T is homogeneous of degree $2d$, and squarefree. Thus the curve is nonsingular of genus $d - 1$.

`long hyperell_locally_soluble(GEN T, GEN p)` assumes that $T \in \mathbf{Z}[X]$ is integral. Returns 1 if the curve is locally soluble over $\mathbf{Q}_p$, 0 otherwise.

`long nf_hyperell_locally_soluble(GEN nf, GEN T, GEN pr)` let K be a number field, associated to `nf`, `pr` a *prid* associated to some maximal ideal $\mathfrak{p}$; assumes that $T \in \mathbf{Z}_K[X]$ is integral. Returns 1 if the curve is locally soluble over $K_\mathfrak{p}$.

Appendix A:
A Sample program and Makefile

We assume that you have installed the PARI library and include files as explained in Appendix A or in the installation guide. If you chose differently any of the directory names, change them accordingly in the Makefiles.

If the program example that we have given is in the file `extgcd.c`, then a sample Makefile might look as follows. Note that the actual file `examples/Makefile` is more elaborate and you should have a look at it if you intend to use `install()` on custom made functions, see Section **??**.

```
CC = cc
INCDIR = /usr/pkg/include
LIBDIR = /usr/pkg/lib
CFLAGS = -O -I$(INCDIR) -L$(LIBDIR)

all:  extgcd

extgcd:  extgcd.c
      $(CC) $(CFLAGS) -o extgcd extgcd.c -lpari -lm
```

We then give the listing of the program `examples/extgcd.c` seen in detail in Section 4.10.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;

  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;

    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviiexact( subii(a, mulii(A,ux)), B );
  gerepileall(av, 3, &a, U, V); return a;
}

int
```

```
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

Appendix B:
PARI and threads

To use PARI in multi-threaded programs, you must configure it using `Configure --enable-tls`. Your system must implement the `__thread` storage class. As a major side effect, this breaks the `libpari` ABI: the resulting library is not compatible with the old one, and `-tls` is appended to the PARI library `soname`. On the other hand, this library is now thread-safe.

PARI provides some functions to set up PARI subthreads. In our model, each concurrent thread needs its own PARI stack. The following scheme is used:

Child thread:

```
void *child_thread(void *arg)
{
  GEN data = pari_thread_start((struct pari_thread*)arg);
  GEN result = ...; /* Compute result from data */
  pari_thread_close();
  return (void*)result;
}
```

Parent thread:

```
  pthread_t th;
  struct pari_thread pth;
  GEN data, result;

  pari_thread_alloc(&pth, s, data);
  pthread_create(&th, NULL, &child_thread, (void*)&pth); /* start child */
  ... /* do stuff in parent */
  pthread_join(th, (void*)&result); /* wait until child terminates */
  result = gcopy(result); /* copy result from thread stack to main stack */
  pari_thread_free(&pth); /* ... and clean up */
```

`void pari_thread_alloc(struct pari_thread *pth, size_t s, GEN arg)` Allocate a PARI stack of size `s` and associate it, together with the argument `arg`, with the PARI thread data `pth`.

`void pari_thread_free(struct pari_thread *pth)` Free the PARI stack associated with the PARI thread data `pth`. This is called after the child thread terminates, i.e. after `pthread_join` in the parent. Any `GEN` objects returned by the child in the thread stack need to be saved before running this command.

`void pari_thread_init(void)` Initialize the thread-local PARI data structures. This function is called by `pari_thread_start`.

`GEN pari_thread_start(struct pari_thread *t)` Initialize the thread-local PARI data structures and set up the thread stack using the PARI thread data `pth`. This function returns the thread argument `arg` that was given to `pari_thread_alloc`.

`void pari_thread_close(void)` Free the thread-local PARI data structures, but keeping the thread stack, so that a `GEN` returned by the thread remains valid.

Under this model, some PARI states are reset in new threads. In particular

- the random number generator is reset to the starting seed;
- the system stack exhaustion checking code, meant to catch infinite recursions, is disabled (use `pari_stackcheck_init()` to reenable it);
- cached real constants (returned by `mppi`, `mpeuler` and `mplog2`) are not shared between threads and will be recomputed as needed;

The following sample program can be compiled using

```
cc thread.c -o thread.o -lpari -lpthread
```

(Add `-I/-L` paths as necessary.)

```
#include <pari/pari.h> /* Include PARI headers */
#include <pthread.h>   /* Include POSIX threads headers */

void *
mydet(void *arg)
{
  GEN F, M;
  /* Set up thread stack and get thread parameter */
  M = pari_thread_start((struct pari_thread*) arg);
  F = det(M);
  /* Free memory used by the thread */
  pari_thread_close();
  return (void*)F;
}

void *
myfactor(void *arg)  /* same principle */
{
  GEN F, N;
  N = pari_thread_start((struct pari_thread*) arg);
  F = factor(N);
  pari_thread_close();
  return (void*)F;
}

int
main(void)
{
  GEN M,N1,N2, F1,F2,D;
  pthread_t th1, th2, th3; /* POSIX-thread variables */
  struct pari_thread pth1, pth2, pth3; /* pari thread variables */

  /* Initialise the main PARI stack and global objects (gen_0, etc.) */
  pari_init(4000000,500000);
  /* Compute in the main PARI stack */
  N1 = addis(int2n(256), 1); /* 2^256 + 1 */
```

```
  N2 = subis(int2n(193), 1); /* 2^193 - 1 */
  M = mathilbert(80);
  /* Allocate pari thread structures */
  pari_thread_alloc(&pth1,4000000,N1);
  pari_thread_alloc(&pth2,4000000,N2);
  pari_thread_alloc(&pth3,4000000,M);
  /* pthread_create() and pthread_join() are standard POSIX-thread
   * functions to start and get the result of threads. */
  pthread_create(&th1,NULL, &myfactor, (void*)&pth1);
  pthread_create(&th2,NULL, &myfactor, (void*)&pth2);
  pthread_create(&th3,NULL, &mydet,    (void*)&pth3); /* Start 3 threads */
  pthread_join(th1,(void*)&F1);
  pthread_join(th2,(void*)&F2);
  pthread_join(th3,(void*)&D); /* Wait for termination, get the results */
  pari_printf("F1=%Ps\nF2=%Ps\nlog(D)=%Ps\n", F1, F2, glog(D,3));
  pari_thread_free(&pth1);
  pari_thread_free(&pth2);
  pari_thread_free(&pth3); /* clean up */
  return 0;
}
```

Index

SomeWord refers to PARI-GP concepts.
`SomeWord` is a PARI-GP keyword.
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