

# PARI-GP Reference Card

(PARI-GP version 2.3.0)

Note: optional arguments are surrounded by braces {}.

## Starting & Stopping GP

to enter GP, just type its name: `gp`  
to exit GP, type `\q` or `quit`

## Help

describe function `?function`  
extended description `??keyword`  
list of relevant help topics `???pattern`

## Input/Output & Defaults

output previous line, the lines before `%, %', %'', etc.`  
output from line  $n$  `%n`  
separate multiple statements on line `;`  
extend statement on additional lines `\`  
extend statements on several lines `{seq1; seq2;`  
comment `/* ... */`  
one-line comment, rest of line ignored `\\ ...`  
set default  $d$  to  $val$  `default({d},{val},flag)`  
mimic behaviour of GP 1.39 `default(compatible,3)`

## Metacommands

toggle timer on/off `#`  
print time for last result `##`  
print  $%n$  in raw format `\a n`  
print  $%n$  in pretty format `\b n`  
print defaults `\d`  
set debug level to  $n$  `\g n`  
set memory debug level to  $n$  `\gm n`  
enable/disable logfile `\l {filename}`  
print  $%n$  in pretty matrix format `\m`  
set output mode (raw, default, prettyprint) `\o n`  
set  $n$  significant digits `\p n`  
set  $n$  terms in series `\ps n`  
quit GP `\q`  
print the list of PARI types `\t`  
print the list of user-defined functions `\u`  
read file into GP `\r filename`  
write  $%n$  to file `\w n filename`

## GP Within Emacs

to enter GP from within Emacs: `M-x gp, C-u M-x gp`  
word completion `(TAB)`  
help menu window `M-\c`  
describe function `M-?`  
display  $\mathrm{T\!E\!X}$ 'd PARI manual `M-x gpman`  
set prompt string `M-\p`  
break line at column 100, insert `M-\\`  
PARI metacommand `\letter` `M-\letter`

## Reserved Variable Names

$\pi = 3.14159\dots$  `Pi`  
Euler's constant  $= .57721\dots$  `Euler`  
square root of  $-1$  `I`  
big-oh notation `O`

## PARI Types & Input Formats

<code>t_INT</code> . Integers	$\pm n$
<code>t_REAL</code> . Real Numbers	$\pm n.ddd$
<code>t_INTMOD</code> . Integers modulo $m$	<code>Mod(<math>n, m</math>)</code>
<code>t_FRAC</code> . Rational Numbers	$n/m$
<code>t_COMPLEX</code> . Complex Numbers	$x + y * I$
<code>t_PADIC</code> . $p$ -adic Numbers	$x + O(p^k)$
<code>t_QUAD</code> . Quadratic Numbers	$x + y * \text{quadgen}(D)$
<code>t_POLMOD</code> . Polynomials modulo $g$	<code>Mod(<math>f, g</math>)</code>
<code>t_POL</code> . Polynomials	$a * x^n + \dots + b$
<code>t_SER</code> . Power Series	$f + O(x^k)$
<code>t_QFI/t_QFR</code> . Imag/Real bin. quad. forms	<code>Qfb(<math>a, b, c, \{d\}</math>)</code>
<code>t_RFRAC</code> . Rational Functions	$f/g$
<code>t_VEC/t_COL</code> . Row/Column Vectors	$[x, y, z], [x, y, z] \sim$
<code>t_MAT</code> . Matrices	$[x, y; z, t; u, v]$
<code>t_LIST</code> . Lists	<code>List(<math>[x, y, z]</math>)</code>
<code>t_STR</code> . Strings	"aaa"

## Standard Operators

basic operations	$+, -, *, /, ^$
$i=i+1, i=i-1, i=i*j, \dots$	$i++, i--, i*=j, \dots$
euclidean quotient, remainder	$x \backslash y, x \backslash y, x \% y, \text{divrem}(x, y)$
shift $x$ left or right $n$ bits	$x << n, x >> n$ or <code>shift(<math>x, n</math>)</code>
comparison operators	$<=, <, >=, >, ==, !=$
boolean operators (or, and, not)	$  , \&\&, !$
sign of $x = -1, 0, 1$	<code>sign(<math>x</math>)</code>
maximum/minimum of $x$ and $y$	<code>max, min(<math>x, y</math>)</code>
integer or real factorial of $x$	$x!$ or <code>factorial(<math>x</math>)</code>
derivative of $f$ w.r.t. $x$	$f'$

## Conversions

### Change Objects

to vector, matrix, set, list, string	<code>Col/Vec, Mat, Set, List, Str</code>
create PARI object ( $x \bmod y$ )	<code>Mod(<math>x, y</math>)</code>
make $x$ a polynomial of $v$	<code>Pol(<math>x, \{v\}</math>)</code>
as above, starting with constant term	<code>Polrev(<math>x, \{v\}</math>)</code>
make $x$ a power series of $v$	<code>Ser(<math>x, \{v\}</math>)</code>
PARI type of object $x$	<code>type(<math>x, \{t\}</math>)</code>
object $x$ with precision $n$	<code>prec(<math>x, \{n\}</math>)</code>
evaluate $f$ replacing vars by their value	<code>eval(<math>f</math>)</code>

### Select Pieces of an Object

length of $x$	<code>#x</code> or <code>length(<math>x</math>)</code>
$n$ -th component of $x$	<code>component(<math>x, n</math>)</code>
$n$ -th component of vector/list $x$	$x[n]$
$(m, n)$ -th component of matrix $x$	$x[m, n]$
row $m$ or column $n$ of matrix $x$	$x[m, ], x[ , n]$
numerator of $x$	<code>numerator(<math>x</math>)</code>
lowest denominator of $x$	<code>denominator(<math>x</math>)</code>

### Conjugates and Lifts

conjugate of a number $x$	<code>conj(<math>x</math>)</code>
conjugate vector of algebraic number $x$	<code>conjvec(<math>x</math>)</code>
norm of $x$ , product with conjugate	<code>norm(<math>x</math>)</code>
square of $L^2$ norm of vector $x$	<code>norml2(<math>x</math>)</code>
lift of $x$ from Mods	<code>lift, centerlift(<math>x</math>)</code>

## Random Numbers

random integer between 0 and  $N - 1$  `random( $\{N\}$ )`  
get random seed `getrand()`  
set random seed to  $s$  `setrand( $s$ )`

## Lists, Sets & Sorting

sort  $x$  by  $k$ th component `vecsort( $x, \{k\}, \{fl = 0\}$ )`  
**Sets** (= row vector of strings with strictly increasing entries)  
intersection of sets  $x$  and  $y$  `setintersect( $x, y$ )`  
set of elements in  $x$  not belonging to  $y$  `setminus( $x, y$ )`  
union of sets  $x$  and  $y$  `setunion( $x, y$ )`  
look if  $y$  belongs to the set  $x$  `setsearch( $x, y, flag$ )`  
**Lists**  
create empty list of maximal length  $n$  `listcreate( $n$ )`  
delete all components of list  $l$  `listkill( $l$ )`  
append  $x$  to list  $l$  `listput( $l, x, \{i\}$ )`  
insert  $x$  in list  $l$  at position  $i$  `listinsert( $l, x, i$ )`  
sort the list  $l$  `listsort( $l, flag$ )`

## Programming & User Functions

**Control Statements** ( $X$ : formal parameter in expression  $seq$ )  
eval.  $seq$  for  $a \leq X \leq b$  `for( $X = a, b, seq$ )`  
eval.  $seq$  for  $X$  dividing  $n$  `fordiv( $n, X, seq$ )`  
eval.  $seq$  for primes  $a \leq X \leq b$  `forprime( $X = a, b, seq$ )`  
eval.  $seq$  for  $a \leq X \leq b$  stepping  $s$  `forstep( $X = a, b, s, seq$ )`  
multivariable for `forvec( $X = v, seq$ )`  
if  $a \neq 0$ , evaluate  $seq_1$ , else  $seq_2$  `if( $a, \{seq_1\}, \{seq_2\}$ )`  
evaluate  $seq$  until  $a \neq 0$  `until( $a, seq$ )`  
while  $a \neq 0$ , evaluate  $seq$  `while( $a, seq$ )`  
exit  $n$  innermost enclosing loops `break( $\{n\}$ )`  
start new iteration of  $n$ th enclosing loop `next( $\{n\}$ )`  
return  $x$  from current subroutine `return( $x$ )`  
error recovery (try  $seq_1$ ) `trap( $\{err\}, \{seq_2\}, \{seq_1\}$ )`  
**Input/Output**  
prettyprint args with/without newline `printp(), printp1()`  
print args with/without newline `print(), print1()`  
read a string from keyboard `input()`  
reorder priority of variables  $x, y, z$  `reorder( $\{[x, y, z]\}$ )`  
output  $args$  in  $\mathrm{T\!E\!X}$  format `printtex( $args$ )`  
write  $args$  to file `write, write1, writetex( $file, args$ )`  
read file into GP `read( $\{file\}$ )`

### Interface with User and System

allocates a new stack of  $s$  bytes `allocatemem( $\{s\}$ )`  
execute system command  $a$  `system( $a$ )`  
as above, feed result to GP `extern( $a$ )`  
install function from library `install( $f, code, \{gpf\}, \{lib\}$ )`  
alias  $old$  to  $new$  `alias( $new, old$ )`  
new name of function  $f$  in GP 2.0 `whatnow( $f$ )`

### User Defined Functions

`name(formal vars) = local(local vars); seq`  
`struct.member = seq`  
kill value of variable or function  $x$  `kill( $x$ )`  
declare global variables `global( $x, \dots$ )`

## Iterations, Sums & Products

numerical integration `intnum( $X = a, b, expr, flag$ )`  
sum  $expr$  over divisors of  $n$  `sumdiv( $n, X, expr$ )`  
sum  $X = a$  to  $X = b$ , initialized at  $x$  `sum( $X = a, b, expr, \{x\}$ )`  
sum of series  $expr$  `suminf( $X = a, expr$ )`  
sum of alternating/positive series `sumalt, sumpos`  
product  $a \leq X \leq b$ , initialized at  $x$  `prod( $X = a, b, expr, \{x\}$ )`  
product over primes  $a \leq X \leq b$  `prodeuler( $X = a, b, expr$ )`  
infinite product  $a \leq X \leq \infty$  `prodinf( $X = a, expr$ )`  
real root of  $expr$  between  $a$  and  $b$  `solve( $X = a, b, expr$ )`

Vectors & Matrices

dimensions of matrix $x$	<code>matsize(<math>x</math>)</code>
concatenation of $x$ and $y$	<code>concat(<math>x, \{y\}</math>)</code>
extract components of $x$	<code>vecextract(<math>x, y, \{z\}</math>)</code>
transpose of vector or matrix $x$	<code>mattranspose(<math>x</math>)</code> or <code>x-</code>
adjoint of the matrix $x$	<code>matadjoin(<math>x</math>)</code>
eigenvectors of matrix $x$	<code>mateigen(<math>x</math>)</code>
characteristic polynomial of $x$	<code>charpoly(<math>x, \{v\}, flag</math>)</code>
minimal polynomial of $x$	<code>minpoly(<math>x, \{v\}</math>)</code>
trace of matrix $x$	<code>trace(<math>x</math>)</code>

Constructors & Special Matrices

row vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vector(<math>n, \{i\}, \{expr\}</math>)</code>
col. vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vectorv(<math>n, \{i\}, \{expr\}</math>)</code>
matrix $1 \leq i \leq m, 1 \leq j \leq n$	<code>matrix(<math>m, n, \{i\}, \{j\}, \{expr\}</math>)</code>
diagonal matrix whose diag. is $x$	<code>matdiagonal(<math>x</math>)</code>
$n \times n$ identity matrix	<code>matid(<math>n</math>)</code>
Hessenberg form of square matrix $x$	<code>mathess(<math>x</math>)</code>
$n \times n$ Hilbert matrix $H_{ij} = (i + j - 1)^{-1}$	<code>mathilbert(<math>n</math>)</code>
$n \times n$ Pascal triangle $P_{ij} = \binom{i}{j}$	<code>matpascal(<math>n - 1</math>)</code>
companion matrix to polynomial $x$	<code>matcompanion(<math>x</math>)</code>

Gaussian elimination

determinant of matrix $x$	<code>matdet(<math>x, flag</math>)</code>
kernel of matrix $x$	<code>matker(<math>x, flag</math>)</code>
intersection of column spaces of $x$ and $y$	<code>matintersect(<math>x, y</math>)</code>
solve $M * X = B$ ( $M$ invertible)	<code>matsolve(<math>M, B</math>)</code>
as solve, modulo $D$ (col. vector)	<code>matsolvemod(<math>M, D, B</math>)</code>
one sol of $M * X = B$	<code>matinverseimage(<math>M, B</math>)</code>
basis for image of matrix $x$	<code>matimage(<math>x</math>)</code>
supplement columns of $x$ to get basis	<code>mat supplement(<math>x</math>)</code>
rows, cols to extract invertible matrix	<code>matindexrank(<math>x</math>)</code>
rank of the matrix $x$	<code>matrank(<math>x</math>)</code>

Lattices & Quadratic Forms

upper triangular Hermite Normal Form	<code>mathnf(<math>x</math>)</code>
HNF of $x$ where $d$ is a multiple of $\det(x)$	<code>mathnfmod(<math>x, d</math>)</code>
elementary divisors of $x$	<code>matsnf(<math>x</math>)</code>
LLL-algorithm applied to columns of $x$	<code>qflll(<math>x, flag</math>)</code>
like <code>qflll</code> , $x$ is Gram matrix of lattice	<code>qflllgram(<math>x, flag</math>)</code>
LLL-reduced basis for kernel of $x$	<code>matkerint(<math>x</math>)</code>
<b>Z</b> -lattice $\longleftrightarrow$ <b>Q</b> -vector space	<code>matrixqz(<math>x, p</math>)</code>
signature of quad form $t^y * x * y$	<code>qfsign(<math>x</math>)</code>
decomp into squares of $t^y * x * y$	<code>qfgaussred(<math>x</math>)</code>
find up to $m$ sols of $t^y * x * y \leq b$	<code>qfminim(<math>x, b, m</math>)</code>
$v, v[i] :=$ number of sols of $t^y * x * y = i$	<code>qfrep(<math>x, B, flag</math>)</code>
eigenvals/eigenvecs for real symmetric $x$	<code>qfjacobi(<math>x</math>)</code>

Formal & p-adic Series

truncate power series or $p$ -adic number	<code>truncate(<math>x</math>)</code>
valuation of $x$ at $p$	<code>valuation(<math>x, p</math>)</code>
<b>Dirichlet and Power Series</b>	
Taylor expansion around 0 of $f$ w.r.t. $x$	<code>taylor(<math>f, x</math>)</code>
$\sum a_k b_k t^k$ from $\sum a_k t^k$ and $\sum b_k t^k$	<code>serconvol(<math>x, y</math>)</code>
$f = \sum a_k * t^k$ from $\sum (a_k / k!) * t^k$	<code>serlaplace(<math>f</math>)</code>
reverse power series $F$ so $F(f(x)) = x$	<code>serreverse(<math>f</math>)</code>
Dirichlet series multiplication / division	<code>dirmul, dirdiv(<math>x, y</math>)</code>
Dirichlet Euler product ( $b$ terms)	<code>direuler(<math>p = a, b, expr</math>)</code>

p-adic Functions

Teichmuller character of $x$	<code>teichmuller(<math>x</math>)</code>
Newton polygon of $f$ for prime $p$	<code>newtonpoly(<math>f, p</math>)</code>

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Polynomials & Rational Functions

degree of $f$	<code>poldegree(<math>f</math>)</code>
coefficient of degree $n$ of $f$	<code>polcoeff(<math>f, n</math>)</code>
round coeffs of $f$ to nearest integer	<code>round(<math>f, \{&amp;e\}</math>)</code>
gcd of coefficients of $f$	<code>content(<math>f</math>)</code>
replace $x$ by $y$ in $f$	<code>subst(<math>f, x, y</math>)</code>
discriminant of polynomial $f$	<code>poldisc(<math>f</math>)</code>
resultant of $f$ and $g$	<code>polresultant(<math>f, g, flag</math>)</code>
as above, give $[u, v, d], xu + yv = d$	<code>bezoutres(<math>x, y</math>)</code>
derivative of $f$ w.r.t. $x$	<code>deriv(<math>f, x</math>)</code>
formal integral of $f$ w.r.t. $x$	<code>intformal(<math>f, x</math>)</code>
reciprocal poly $x^{\deg f} f(1/x)$	<code>polrecip(<math>f</math>)</code>
interpol. pol. eval. at $a$	<code>polinterpolate(<math>X, \{Y\}, \{a\}, \{&amp;e\}</math>)</code>
initialize $t$ for Thue equation solver	<code>thueinit(<math>f</math>)</code>
solve Thue equation $f(x, y) = a$	<code>thue(<math>t, a, \{sol\}</math>)</code>

Roots and Factorization

number of real roots of $f, a < x \leq b$	<code>polsturm(<math>f, \{a\}, \{b\}</math>)</code>
complex roots of $f$	<code>polroots(<math>f</math>)</code>
symmetric powers of roots of $f$ up to $n$	<code>polsym(<math>f, n</math>)</code>
roots of $f$ mod $p$	<code>polrootsmod(<math>f, p, flag</math>)</code>
factor $f$	<code>factor(<math>f, \{lim\}</math>)</code>
factorization of $f$ mod $p$	<code>factormod(<math>f, p, flag</math>)</code>
factorization of $f$ over $\mathbf{F}_{p^a}$	<code>factorff(<math>f, p, a</math>)</code>
$p$ -adic fact. of $f$ to prec. $r$	<code>factorpadic(<math>f, p, r, flag</math>)</code>
$p$ -adic roots of $f$ to prec. $r$	<code>polrootspadic(<math>f, p, r</math>)</code>
$p$ -adic root of $f$ cong. to $a$ mod $p$	<code>padicappr(<math>f, a</math>)</code>
Newton polygon of $f$ for prime $p$	<code>newtonpoly(<math>f, p</math>)</code>

Special Polynomials

$n$ th cyclotomic polynomial in var. $v$	<code>polcyclo(<math>n, \{v\}</math>)</code>
$d$ -th degree subfield of $\mathbf{Q}(\zeta_n)$	<code>polsubcyclo(<math>n, d, \{v\}</math>)</code>
$n$ -th Legendre polynomial	<code>pollegendre(<math>n</math>)</code>
$n$ -th Tchebicheff polynomial	<code>poltchebi(<math>n</math>)</code>
Zagier's polynomial of index $n, m$	<code>polzagier(<math>n, m</math>)</code>

Transcendental Functions

real, imaginary part of $x$	<code>real(<math>x</math>), imag(<math>x</math>)</code>
absolute value, argument of $x$	<code>abs(<math>x</math>), arg(<math>x</math>)</code>
square/ $n$ th root of $x$	<code>sqrtn(<math>x, n, &amp;z</math>)</code>
trig functions	<code>sin, cos, tan, cotan</code>
inverse trig functions	<code>asin, acos, atan</code>
hyperbolic functions	<code>sinh, cosh, tanh</code>
inverse hyperbolic functions	<code>asinh, acosh, atanh</code>
exponential of $x$	<code>exp(<math>x</math>)</code>
natural log of $x$	<code>ln(<math>x</math>)</code> or <code>log(<math>x</math>)</code>
gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$	<code>gamma(<math>x</math>)</code>
logarithm of gamma function	<code>lngamma(<math>x</math>)</code>
$\psi(x) = \Gamma'(x) / \Gamma(x)$	<code>psi(<math>x</math>)</code>
incomplete gamma function ( $y = \Gamma(s)$ )	<code>incgam(<math>s, x, \{y\}</math>)</code>
exponential integral $\int_x^\infty e^{-t} / t dt$	<code>eint1(<math>x</math>)</code>
error function $2 / \sqrt{\pi} \int_x^\infty e^{-t^2} dt$	<code>erfc(<math>x</math>)</code>
dilogarithm of $x$	<code>dilog(<math>x</math>)</code>
$m$ th polylogarithm of $x$	<code>polylog(<math>m, x, flag</math>)</code>
$U$ -confluent hypergeometric function	<code>hyperu(<math>a, b, u</math>)</code>
$J$ -Bessel function $J_{n+1/2}(x)$	<code>besseljh(<math>n, x</math>)</code>
$K$ -Bessel function of index $nu$	<code>besselk(<math>nu, x</math>)</code>

Elementary Arithmetic Functions

vector of binary digits of $ x $	<code>binary(<math>x</math>)</code>
give bit number $n$ of integer $x$	<code>bittest(<math>x, n</math>)</code>
ceiling of $x$	<code>ceil(<math>x</math>)</code>
floor of $x$	<code>floor(<math>x</math>)</code>
fractional part of $x$	<code>frac(<math>x</math>)</code>
round $x$ to nearest integer	<code>round(<math>x, \{&amp;e\}</math>)</code>
truncate $x$	<code>truncate(<math>x, \{&amp;e\}</math>)</code>
gcd/LCM of $x$ and $y$	<code>gcd(<math>x, y</math>), lcm(<math>x, y</math>)</code>
gcd of entries of a vector/matrix	<code>content(<math>x</math>)</code>

Primes and Factorization

add primes in $v$ to the prime table	<code>addprimes(<math>v</math>)</code>
the $n$ th prime	<code>prime(<math>n</math>)</code>
vector of first $n$ primes	<code>primes(<math>n</math>)</code>
smallest prime $\geq x$	<code>nextprime(<math>x</math>)</code>
largest prime $\leq x$	<code>precprime(<math>x</math>)</code>
factorization of $x$	<code>factor(<math>x, \{lim\}</math>)</code>
reconstruct $x$ from its factorization	<code>factorback(<math>fa, \{nf\}</math>)</code>

Divisors

number of distinct prime divisors	<code>omega(<math>x</math>)</code>
number of prime divisors with mult	<code>bigomega(<math>x</math>)</code>
number of divisors of $x$	<code>numdiv(<math>x</math>)</code>
row vector of divisors of $x$	<code>divisors(<math>x</math>)</code>
sum of ( $k$ -th powers of) divisors of $x$	<code>sigma(<math>x, \{k\}</math>)</code>

Special Functions and Numbers

binomial coefficient $\binom{x}{y}$	<code>binomial(<math>x, y</math>)</code>
Bernoulli number $B_n$ as real	<code>bernreal(<math>n</math>)</code>
Bernoulli vector $B_0, B_2, \dots, B_{2n}$	<code>bernvec(<math>n</math>)</code>
$n$ th Fibonacci number	<code>fibonacci(<math>n</math>)</code>
number of partitions of $n$	<code>numbpart(<math>n</math>)</code>
Euler $\phi$ -function	<code>eulerphi(<math>x</math>)</code>
Möbius $\mu$ -function	<code>moebius(<math>x</math>)</code>
Hilbert symbol of $x$ and $y$ (at $p$ )	<code>hilbert(<math>x, y, \{p\}</math>)</code>
Kronecker-Legendre symbol $(\frac{x}{y})$	<code>kronecker(<math>x, y</math>)</code>

Miscellaneous

integer or real factorial of $x$	<code>x!</code> or <code>fact(<math>x</math>)</code>
integer square root of $x$	<code>sqrntint(<math>x</math>)</code>
solve $z \equiv x$ and $z \equiv y$	<code>chinese(<math>x, y</math>)</code>
minimal $u, v$ so $xu + yv = \gcd(x, y)$	<code>bezout(<math>x, y</math>)</code>
multiplicative order of $x$ (intmod) (i=0)	<code>znorder(<math>x, \{o\}</math>)</code>
primitive root mod prime power $q$	<code>znprimroot(<math>q</math>)</code>
structure of $(\mathbf{Z}/n\mathbf{Z})^*$	<code>znstar(<math>n</math>)</code>
continued fraction of $x$	<code>contfrac(<math>x, \{b\}, \{lmax\}</math>)</code>
last convergent of continued fraction $x$	<code>contfracpnqn(<math>x</math>)</code>
best rational approximation to $x$	<code>bestappr(<math>x, k</math>)</code>

True-False Tests

is $x$ the disc. of a quadratic field?	<code>isfundamental(<math>x</math>)</code>
is $x$ a prime?	<code>isprime(<math>x</math>)</code>
is $x$ a strong pseudo-prime?	<code>ispseudoprime(<math>x</math>)</code>
is $x$ square-free?	<code>issquarefree(<math>x</math>)</code>
is $x$ a square?	<code>Z_issquare(<math>x, \{&amp;n\}</math>)</code>
is $pol$ irreducible?	<code>polisirreducible(<math>pol</math>)</code>

Based on an earlier version by Joseph H. Silverman

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Send comments and corrections to <Karim.BELABAS@math.u-psud.fr>

# PARI-GP Reference Card (2)

(PARI-GP version 2.3.0)

## Elliptic Curves

Elliptic curve initially given by 5-tuple  $E = [a_1, a_2, a_3, a_4, a_6]$ . Points are  $[x, y]$ , the origin is  $[0]$ .

Initialize elliptic struct.  $ell$ , i.e create `ellinit( $E, flag$ )`

$a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6, b_8, c_4, c_6, disc, j$ . This data can be recovered by typing  $ell.a1, \dots, ell.j$ . If  $fl$  omitted, also

$E$  defined over **R**

$x$ -coords. of points of order 2	<code>ell.roots</code>
real and complex periods	<code>ell.omega</code>
associated quasi-periods	<code>ell.eta</code>
volume of complex lattice	<code>ell.area</code>

$E$  defined over  $\mathbf{Q}_p, |j|_p > 1$

$x$ -coord. of unit 2 torsion point	<code>ell.roots</code>
Tate's $[u^2, u, q]$	<code>ell.tate</code>
Mestre's $w$	<code>ell.w</code>

change curve  $E$  using  $v = [u, r, s, t]$  `ellchangecurve( $ell, v$ )`

change point  $z$  using  $v = [u, r, s, t]$  `ellchangepoint( $z, v$ )`

cond, min mod, Tamagawa num  $[N, v, c]$  `ellglobalred( $ell$ )`

Kodaira type of  $p$  fiber of  $E$  `elllocalred( $ell, p$ )`

add points  $z_1 + z_2$  `elladd( $ell, z_1, z_2$ )`

subtract points  $z_1 - z_2$  `ellsub( $ell, z_1, z_2$ )`

compute  $n \cdot z$  `ellpow( $ell, z, n$ )`

check if  $z$  is on  $E$  `ellisoncurve( $ell, z$ )`

order of torsion point  $z$  `ellorder( $ell, z$ )`

torsion subgroup with generators `elltors( $ell$ )`

$y$ -coordinates of point(s) for  $x$  `ellordinate( $ell, x$ )`

canonical bilinear form taken at  $z_1, z_2$  `ellbil( $ell, z_1, z_2$ )`

canonical height of  $z$  `ellheight( $ell, z, flag$ )`

height regulator matrix for pts in  $x$  `ellheightmatrix( $ell, x$ )`

$p$ th coeff  $a_p$  of  $L$ -function,  $p$  prime `ellap( $ell, p$ )`

$k$ th coeff  $a_k$  of  $L$ -function `ellak( $ell, k$ )`

vector of first  $n$   $a_k$ 's in  $L$ -function `ellan( $ell, n$ )`

$L(E, s)$ , set  $A \approx 1$  `elllseries( $ell, s, \{A\}$ )`

root number for  $L(E, \cdot)$  at  $p$  `ellrootno( $ell, \{p\}$ )`

modular parametrization of  $E$  `elltaniyama( $ell$ )`

point  $[\wp(z), \wp'(z)]$  corresp. to  $z$  `ellztopoint( $ell, z$ )`

complex  $z$  such that  $p = [\wp(z), \wp'(z)]$  `ellpointtoz( $ell, p$ )`

## Elliptic & Modular Functions

arithmetic-geometric mean `agm( $x, y$ )`

elliptic  $j$ -function  $1/q + 744 + \dots$  `ellj( $x$ )`

Weierstrass  $\sigma$  function `ellsigma( $ell, z, flag$ )`

Weierstrass  $\wp$  function `ellwp( $ell, \{z\}, flag$ )`

Weierstrass  $\zeta$  function `ellzeta( $ell, z$ )`

modified Dedekind  $\eta$  func.  $\prod(1 - q^n)$  `eta( $x, flag$ )`

Jacobi sine theta function `theta( $q, z$ )`

k-th derivative at  $z=0$  of `thetanullk( $q, k$ )`

Weber's  $f$  functions `weber( $x, flag$ )`

Riemann's zeta  $\zeta(s) = \sum n^{-s}$  `zeta( $s$ )`

## Graphic Functions

crude graph of  $expr$  between  $a$  and  $b$  `plot( $X = a, b, expr$ )`

**High-resolution plot** (immediate plot)

plot  $expr$  between  $a$  and  $b$  `plotth( $X = a, b, expr, flag, \{n\}$ )`

plot points given by lists  $lx, ly$  `plotthraw( $lx, ly, flag$ )`

terminal dimensions `plotsizes()`

### Rectwindow functions

init window  $w$ , with size  $x, y$  `plotinit( $w, x, y$ )`

erase window  $w$  `plotkill( $w$ )`

copy  $w$  to  $w_2$  with offset  $(dx, dy)$  `plotcopy( $w, w_2, dx, dy$ )`

scale coordinates in  $w$  `plotscale( $w, x_1, x_2, y_1, y_2$ )`

`plotth` in  $w$  `plotrecth( $w, X = a, b, expr, flag, \{n\}$ )`

`plotthraw` in  $w$  `plotrecthraw( $w, data, flag$ )`

draw window  $w_1$  at  $(x_1, y_1), \dots$  `plotdraw( $[[w_1, x_1, y_1], \dots]$ )`

### Low-level Rectwindow Functions

set current drawing color in  $w$  to  $c$  `plotcolor( $w, c$ )`

current position of cursor in  $w$  `plotcursor( $w$ )`

write  $s$  at cursor's position `plotstring( $w, s$ )`

move cursor to  $(x, y)$  `plotmove( $w, x, y$ )`

move cursor to  $(x + dx, y + dy)$  `plotrmove( $w, dx, dy$ )`

draw a box to  $(x_2, y_2)$  `plotbox( $w, x_2, y_2$ )`

draw a box to  $(x + dx, y + dy)$  `plotrbox( $w, dx, dy$ )`

draw polygon `plotlines( $w, lx, ly, flag$ )`

draw points `plotpoints( $w, lx, ly$ )`

draw line to  $(x + dx, y + dy)$  `plotrline( $w, dx, dy$ )`

draw point  $(x + dx, y + dy)$  `plotrpoint( $w, dx, dy$ )`

### Postscript Functions

as `plotth` `psplotth( $X = a, b, expr, flag, \{n\}$ )`

as `plotthraw` `psplotthraw( $lx, ly, flag$ )`

as `plotdraw` `psdraw( $[[w_1, x_1, y_1], \dots]$ )`

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ ) `qfb( $a, b, c, \{d\}$ )`

reduce  $x$  ( $s = \sqrt{D}, l = \lfloor s \rfloor$ ) `qfbred( $x, flag, \{D\}, \{l\}, \{s\}$ )`

composition of forms  $x*y$  or `qfbnucomp( $x, y, l$ )`

$n$ -th power of form  $x^n$  or `qfbnupow( $x, n$ )`

composition without reduction `qfbcompraw( $x, y$ )`

$n$ -th power without reduction `qfbpowraw( $x, n$ )`

prime form of disc.  $x$  above prime  $p$  `qfbprimeform( $x, p$ )`

class number of disc.  $x$  `qfbclassno( $x$ )`

Hurwitz class number of disc.  $x$  `qfbhclassno( $x$ )`

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$  `quadgen( $x$ )`

minimal polynomial of  $\omega$  `quadpoly( $x$ )`

discriminant of  $\mathbf{Q}(\sqrt{D})$  `quaddisc( $x$ )`

regulator of real quadratic field `quadregulator( $x$ )`

fundamental unit in real  $\mathbf{Q}(x)$  `quadunit( $x$ )`

class group of  $\mathbf{Q}(\sqrt{D})$  `quadclassunit( $D, flag, \{t\}$ )`

Hilbert class field of  $\mathbf{Q}(\sqrt{D})$  `quadhilbert( $D, flag$ )`

ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$  `quadrday( $D, f, flag$ )`

## General Number Fields: Initializations

A number field  $K$  is given by a monic irreducible  $f \in \mathbf{Z}[X]$ .

init number field structure  $nf$  `nfinit( $f, flag$ )`

### nf members:

polynomial defining $nf$ , $f(\theta) = 0$	<code>nf.pol</code>
number of real/complex places	<code>nf.r1, nf.r2</code>
discriminant of $nf$	<code>nf.disc</code>
$T_2$ matrix	<code>nf.t2</code>
vector of roots of $f$	<code>nf.roots</code>
integral basis of $\mathbf{Z}_K$ as powers of $\theta$	<code>nf.zk</code>
different	<code>nf.diff</code>
codifferent	<code>nf.codiff</code>

recompute  $nf$  using current precision `nfnewprec( $nf$ )`

init relative  $rmf$  given by  $g = 0$  over  $K$  `rmfinit( $nf, g$ )`

init  $bnf$  structure `bnfinit( $f, flag$ )`

**bnf members:** same as  $nf$ , plus

underlying $nf$	<code>bnf.nf</code>
classgroup	<code>bnf.clgp</code>
regulator	<code>bnf.reg</code>
fundamental units	<code>bnf.fu</code>
torsion units	<code>bnf.tu</code>
$[tu, fu]$	<code>bnf.tufu</code>

compute a  $bnf$  from small  $bnf$  `bnfmake( $sbnf$ )`

add  $S$ -class group and units, yield  $bnf$  s `bnfsunit( $nf, S$ )`

init class field structure  $bnr$  `bnrinit( $bnf, m, flag$ )`

**bnr members:** same as  $bnf$ , plus

underlying $bnf$	<code>bnr.bnf</code>
structure of $(\mathbf{Z}_K/m)^*$	<code>bnr.zkst</code>

## Simple Arithmetic Invariants (nf)

Elements are rational numbers, polynomials, polmods, or column vectors (on integral basis  $nf.zk$ ).

integral basis of field def. by  $f = 0$       **nfbasis**( $f$ )  
field discriminant of field  $f = 0$       **nfdisc**( $f$ )  
reverse polmod  $a = A(X) \bmod T(X)$       **modreverse**( $a$ )  
Galois group of field  $f = 0$ ,  $\deg f \leq 11$       **polgalois**( $f$ )  
smallest poly defining  $f = 0$       **polredabs**( $f, flag$ )  
small polys defining subfields of  $f = 0$       **polred**( $f, flag, \{p\}$ )  
small polys defining suborders of  $f = 0$       **polredord**( $f$ )  
poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       **algdep**( $x, k$ )  
small linear rel. on coords of vector  $x$       **lindep**( $x$ )  
are fields  $f = 0$  and  $g = 0$  isomorphic?      **nfisom**( $f, g$ )  
is field  $f = 0$  a subfield of  $g = 0$ ?      **nfisincl**( $f, g$ )  
compositum of  $f = 0$ ,  $g = 0$       **polcompositum**( $f, g, flag$ )  
basic element operations (prefix **nfelt**):

(**nfelt**)**mul**, **pow**, **div**, **diveuc**, **mod**, **divrem**, **val**  
express  $x$  on integer basis      **nfalgtobasis**( $nf, x$ )  
express element  $x$  as a polmod      **nfbasistoalg**( $nf, x$ )  
quadratic Hilbert symbol (at  $p$ )      **nfhilbert**( $nf, a, b, \{p\}$ )  
roots of  $g$  belonging to  $nf$       **nfroots**( $\{nf\}, g$ )  
factor  $g$  in  $nf$       **nfactor**( $nf, g$ )  
factor  $g$  mod prime  $pr$  in  $nf$       **nfactormod**( $nf, g, pr$ )  
number of roots of unity in  $nf$       **nfrootsof1**( $nf$ )  
conjugates of a root  $\theta$  of  $nf$       **nfgaloisconj**( $nf, flag$ )  
apply Galois automorphism  $s$  to  $x$       **nfgaloisapply**( $nf, s, x$ )  
subfields (of degree  $d$ ) of  $nf$       **nfsubfields**( $nf, \{d\}$ )

### Dedekind Zeta Function $\zeta_K$

$\zeta_K$  as Dirichlet series,  $N(I) < b$       **dirzetak**( $nf, b$ )  
init  $nfz$  for field  $f = 0$       **zetakinit**( $f$ )  
compute  $\zeta_K(s)$       **zetak**( $nfz, s, flag$ )  
Artin root number of  $K$       **bnrrootnumber**( $bnr, chi, flag$ )

## Class Groups & Units (bnf, bnr)

$a_1, \{a_2\}, \{a_3\}$  usually  $bnr, subgp$  or  $bnf, module, \{subgp\}$   
remove GRH assumption from  $bnf$       **bnfcertify**( $bnf$ )  
expo. of ideal  $x$  on class gp      **bnfisprincipal**( $bnf, x, flag$ )  
expo. of ideal  $x$  on ray class gp      **bnrisprincipal**( $bnr, x, flag$ )  
expo. of  $x$  on fund. units      **bnfisunit**( $bnf, x$ )  
as above for  $S$ -units      **bnfissunit**( $bnfs, x$ )  
fundamental units of  $bnf$       **bnfunit**( $bnf$ )  
signs of real embeddings of  $bnf.fu$       **bnfsignunit**( $bnf$ )

### Class Field Theory

ray class group structure for mod.  $m$       **bnrclass**( $bnf, m, flag$ )  
ray class number for mod.  $m$       **bnrclassno**( $bnf, m$ )  
discriminant of class field ext      **bnrdisc**( $a_1, \{a_2\}, \{a_3\}$ )  
ray class numbers,  $l$  list of mods      **bnrclassnolist**( $bnf, l$ )  
discriminants of class fields      **bnrdisclist**( $bnf, l, \{arch\}, flag$ )  
decode output from **bnrdisc**      **bnfdecodemodule**( $nf, fa$ )  
is modulus the conductor?      **bnrisconductor**( $a_1, \{a_2\}, \{a_3\}$ )  
conductor of character  $chi$       **bnrconductorofchar**( $bnr, chi$ )  
conductor of extension      **bnrconductor**( $a_1, \{a_2\}, \{a_3\}, flag$ )  
conductor of extension def. by  $g$       **rnfconductor**( $bnf, g$ )  
Artin group of ext. def'd by  $g$       **rnfnormgroup**( $bnr, g$ )  
subgroups of  $bnr$ , index  $\leq b$       **subgrouplist**( $bnr, b, flag$ )  
rel. eq. for class field def'd by  $sub$       **rnfkummer**( $bnr, sub, \{d\}$ )  
same, using Stark units (real field)      **bnrstark**( $bnr, sub, flag$ )

## PARI-GP Reference Card (2)

(PARI-GP version 2.3.0)

### Ideals

Ideals are elements, primes, or matrix of generators in HNF.  
is  $id$  an ideal in  $nf$ ?      **nfisideal**( $nf, id$ )  
is  $x$  principal in  $bnf$ ?      **bnfisprincipal**( $bnf, x$ )  
principal ideal generated by  $x$       **idealprincipal**( $nf, x$ )  
principal idele generated by  $x$       **ideleprincipal**( $nf, x$ )  
give  $[a, b]$ , s.t.  $a\mathbf{Z}_K + b\mathbf{Z}_K = x$       **idealtwoelt**( $nf, x, \{a\}$ )  
put ideal  $a$  ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form      **idealhnf**( $nf, a, \{b\}$ )  
norm of ideal  $x$       **idealnrm**( $nf, x$ )  
minimum of ideal  $x$  (direction  $v$ )      **idealmin**( $nf, x, v$ )  
LLL-reduce the ideal  $x$  (direction  $v$ )      **idealred**( $nf, x, \{v\}$ )

### Ideal Operations

add ideals  $x$  and  $y$       **idealadd**( $nf, x, y$ )  
multiply ideals  $x$  and  $y$       **idealmul**( $nf, x, y, flag$ )  
intersection of ideals  $x$  and  $y$       **idealintersect**( $nf, x, y, flag$ )  
 $n$ -th power of ideal  $x$       **idealpow**( $nf, x, n, flag$ )  
inverse of ideal  $x$       **idealinv**( $nf, x$ )  
divide ideal  $x$  by  $y$       **idealdiv**( $nf, x, y, flag$ )  
Find  $(a, b) \in x \times y$ ,  $a + b = 1$       **idealaddtoone**( $nf, x, \{y\}$ )

### Primes and Multiplicative Structure

factor ideal  $x$  in  $nf$       **idealfactor**( $nf, x$ )  
recover  $x$  from its factorization in  $nf$       **factorback**( $x, nf$ )  
decomposition of prime  $p$  in  $nf$       **idealprimedec**( $nf, p$ )  
valuation of  $x$  at prime ideal  $pr$       **idealval**( $nf, x, pr$ )  
weak approximation theorem in  $nf$       **idealchinese**( $nf, x, y$ )  
give  $bid$  = structure of  $(\mathbf{Z}_K/id)^*$       **idealstar**( $nf, id, flag$ )  
discrete log of  $x$  in  $(\mathbf{Z}_K/bid)^*$       **ideallog**( $nf, x, bid$ )  
**idealstar** of all ideals of norm  $\leq b$       **ideallist**( $nf, b, flag$ )  
add archimedean places      **ideallistarch**( $nf, b, \{ar\}, flag$ )  
init **prmod** structure      **nfmodprinit**( $nf, pr$ )  
kernel of matrix  $M$  in  $(\mathbf{Z}_K/pr)^*$       **nfkermodpr**( $nf, M, prmod$ )  
solve  $Mx = B$  in  $(\mathbf{Z}_K/pr)^*$       **nfsolvemodpr**( $nf, M, B, prmod$ )

### Galois theory over $\mathbf{q}$

initializes a Galois group structure      **galoisinit**( $pol, \{den\}$ )  
action of  $p$  in **nfgaloisconj** form      **galoispermopol**( $G, \{p\}$ )  
identifies as abstract group      **galoisidentify**( $G$ )  
exports a group for GAP or MAGMA      **galoisexport**( $G, flag$ )  
subgroups of the Galois group  $G$       **galoissubgroups**( $G$ )  
subfields from subgroups of  $G$       **galoissubfields**( $G, flag, \{v\}$ )  
fixed field      **galoisfixedfield**( $G, perm, flag, \{v\}$ )  
is  $G$  abelian?      **galoisisabelian**( $G, flag$ )  
abelian number fields      **galoissubcyclo**( $N, H, flag, \{v\}$ )

## Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $g \in K[x]$ . We have  $order \subset L$ .  
absolute equation of  $L$       **rnfequation**( $nf, g, flag$ )  
relative **nfalgtobasis**      **rnfalgtobasis**( $rnf, x$ )  
relative **nfbasistoalg**      **rnfbasistoalg**( $rnf, x$ )  
relative **idealhnf**      **rnfidealhnf**( $rnf, x$ )  
relative **idealmul**      **rnfidealmul**( $rnf, x, y$ )  
relative **idealtwoelt**      **rnfidealtwoelt**( $rnf, x$ )

### Lifts and Push-downs

absolute  $\rightarrow$  relative repres. for  $x$       **rnfeltabstorel**( $rnf, x$ )  
relative  $\rightarrow$  absolute repres. for  $x$       **rnfeltreltoabs**( $rnf, x$ )  
lift  $x$  to the relative field      **rnfeltup**( $rnf, x$ )  
push  $x$  down to the base field      **rnfeltdown**( $rnf, x$ )  
idem for  $x$  ideal: (**rnfideal**)**reltoabs**, **abstorel**, **up**, **down**

### Projective $\mathbf{Z}_K$ -modules, maximal order

relative **polred**      **rnfpolred**( $nf, g$ )  
relative **polredabs**      **rnfpolredabs**( $nf, g$ )  
characteristic poly. of  $a \bmod g$       **rnfcharpoly**( $nf, g, a, \{v\}$ )  
relative Dedekind criterion, prime  $pr$       **rnfdedekind**( $nf, g, pr$ )  
discriminant of relative extension      **rnfdisc**( $nf, g$ )  
pseudo-basis of  $\mathbf{Z}_L$       **rnfpseudobasis**( $nf, g$ )  
relative HNF basis of  $order$       **rnfhnfbasis**( $bnf, order$ )  
reduced basis for  $order$       **rnflllgram**( $nf, g, order$ )  
determinant of pseudo-matrix  $A$       **rnfdet**( $nf, A$ )  
Steinitz class of  $order$       **rnfstesinitz**( $nf, order$ )  
is  $order$  a free  $\mathbf{Z}_K$ -module?      **rnfisfree**( $bnf, order$ )  
true basis of  $order$ , if it is free      **rnfbasis**( $bnf, order$ )

### Norms

absolute norm of ideal  $x$       **rnfidealnrmabs**( $rnf, x$ )  
relative norm of ideal  $x$       **rnfidealnrmrel**( $rnf, x$ )  
solutions of  $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$       **bnfisintnorm**( $bnf, x$ )  
is  $x \in \mathbf{Q}$  a norm from  $K$ ?      **bnfisnorm**( $bnf, x, flag$ )  
initialize  $T$  for norm eq. solver      **rnfisnorminit**( $K, pol, flag$ )  
is  $a \in K$  a norm from  $L$ ?      **rnfisnorm**( $T, a, flag$ )

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Send comments and corrections to <Karim.BELABAS@math.u-psud.fr>